# CHAPTER 11
# THE THICK-WALL CYLINDER

## 11.1 BASIC RELATIONS

In this section, we derive basic relations for the axisymmetric deformation of a thick-wall cylinder. Thick-wall cylinders are used widely in industry as pressure vessels, pipes, gun tubes, etc. In many applications the cylinder wall thickness is constant and the cylinder is subjected to a uniform internal pressure $p_1$, a uniform external pressure $p_2$, an axial load $P$, and a temperature change $\Delta T$ (measured from an initial uniform reference temperature; see Section 3.4) (Figure 11.1). Often the temperature change $\Delta T$ is a function of the radial coordinate $r$ only.

Under such conditions, the deformations of the cylinder are symmetrical with respect to the axis of the cylinder (axisymmetric). Furthermore, the deformations at a cross section sufficiently far removed from the junction of the cylinder and its end caps (Figure 11.1) are practically independent of the axial coordinate $z$. In particular, if the cylinder is open (no end caps) and unconstrained, it undergoes axisymmetric deformations owing to pressures $p_1$ and $p_2$ and temperature change $\Delta T = \Delta T(r)$, which are independent of $z$. If the cylinder's deformation is constrained by supports or end caps, then in the vicinity of the supports or junction between the cylinder and end caps, the deformation and stresses will depend on the axial coordinate $z$.

For example, consider a pressure tank formed by welding together hemispherical caps and a cylinder (Figure 11.2). Under the action of an internal pressure $p_1$, the tank deforms as indicated by the dotted inside boundary and the long dashed outside boundary (the deformations are exaggerated in Figure 11.2). If the cylinder were not constrained by the end caps, it would be able to undergo a larger radial displacement. However, at the junctions between the hemispherical caps and cylinder, the cylinder displacement is constrained by the stiff hemispherical caps. Consequently, the radial displacement (and hence the strains and stresses) at cylinder cross sections near the end cap junctions differs from those at sections far removed from the end cap junctions.

In this section, we consider the displacement, strains, and stresses at locations far removed from the end caps. The determination of deformations, strains, and stresses near the junction of the thick-wall end caps and the thick-wall cylinder lies outside the scope of our treatment. This problem often is treated by experimental methods, since its analytical solution depends on a general three-dimensional study in the theory of elasticity (or plasticity). For thin-wall cylinders, the stress near the end cap junctions may be estimated by the procedure outlined in Section 10.7 (see Problem 10.49).

Consequently, the solution presented in this chapter for thick-wall cylinders is applicable to locations sufficiently far from the end cap junctions where the effects of the constraints imposed by the end caps are negligible. The solution is also applicable to thick-wall cylinders that do not have end caps, so-called open cylinders. Since only axially symmetrical loads and constraints are admitted, the solution is axisymmetrical, that is, a function only of radial coordinate $r$.

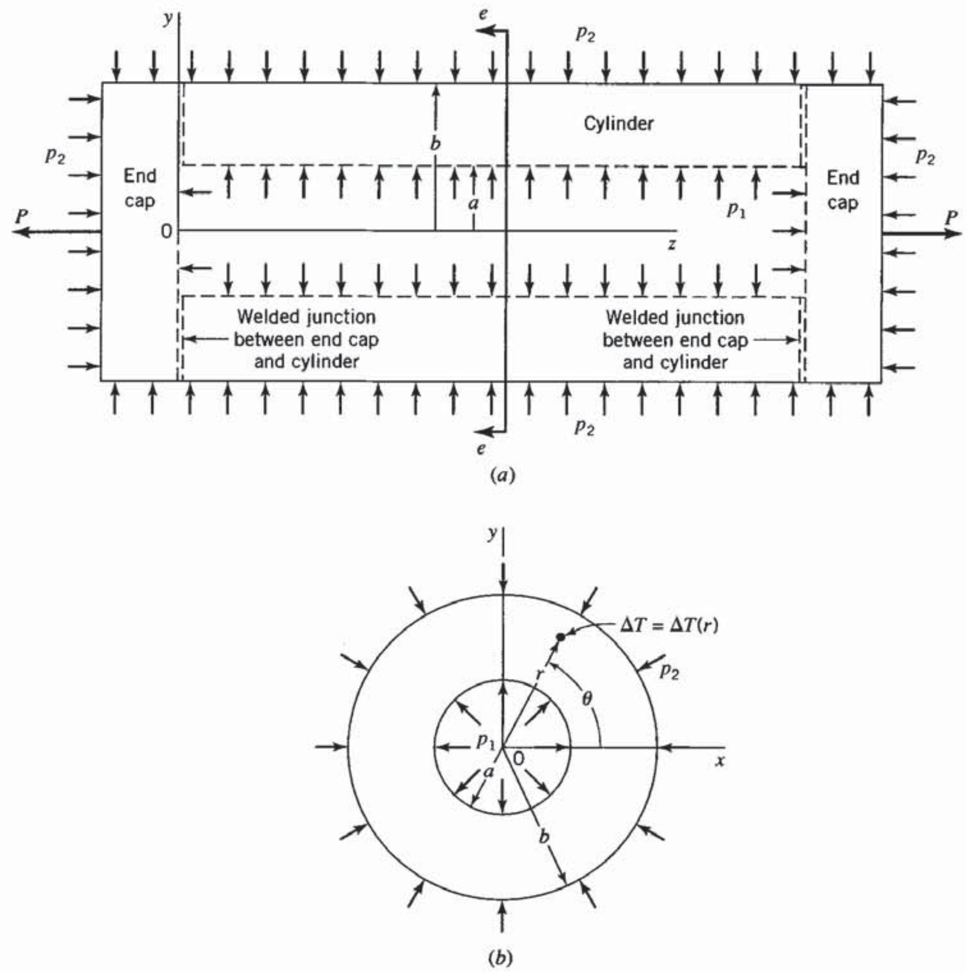


**FIGURE 11.1** Closed cylinder with internal pressure, external pressure, and axial loads. (*a*) Closed cylinder. (*b*) Section *e–e*.

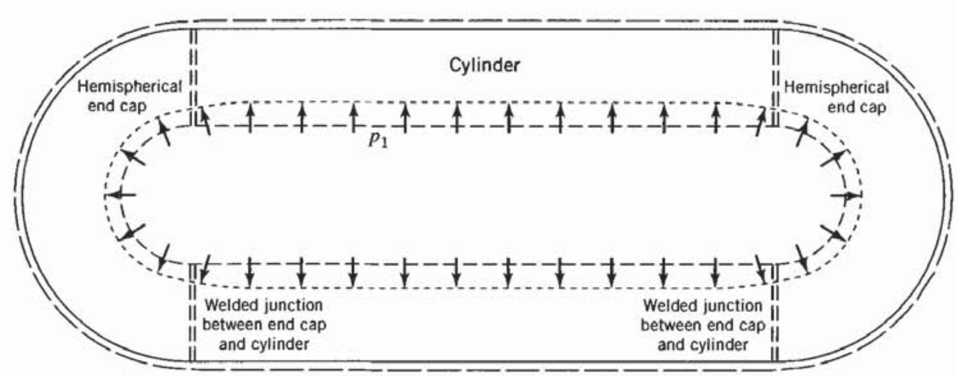


**FIGURE 11.2** Closed cylinder with hemispherical ends.

We use cylindrical coordinates $r$, $\theta$, $z$ for radial, circumferential, and axial directions. Let the cylinder be loaded as shown in Figure 11.1. For analysis purposes, we remove a thin annulus of thickness $dz$ from the cylinder (far removed from the end junctions) by passing two planes perpendicular to the $z$ axis, a distance $dz$ apart (Figure 11.3*a*). The cylindrical volume element $dr\,(r\,d\theta)\,dz$ shown in Figure 11.3*b* is removed from the annulus. Because of radial symmetry, no shear stresses act on the volume element and normal stresses are functions of $r$ only. The nonzero stress components are principal stresses $\sigma_{rr}$, $\sigma_{\theta\theta}$, and $\sigma_{zz}$. The distributions of these stresses through the wall thickness are determined by the equations of equilibrium, compatibility relations, stress–strain–temperature relations, and material response data.

## 11.1.1 Equation of Equilibrium

We neglect body force components. Hence, the equations of equilibrium for cylindrical coordinates (Eqs. 2.50) reduce to the single equation

$$r\frac{d\sigma_{rr}}{dr} = \sigma_{\theta\theta} - \sigma_{rr} \quad \text{or} \quad \frac{d}{dr}(r\sigma_{rr}) = \sigma_{\theta\theta} \tag{11.1}$$

## 11.1.2 Strain–Displacement Relations and Compatibility Condition

The strain–displacement relations for the thick-walled cylinder (Eqs. 2.85) yield the three relations for extensional strains

$$\epsilon_{rr} = \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u}{r}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z} \tag{11.2}$$

where $u = u(r, z)$ and $w = w(r, z)$ denote displacement components in the $r$ and $z$ directions, respectively. At sections far removed from the ends, the dependency on $z$ in $u$ and $w$ is considered to be small. Hence, at sections far from the ends, the shear strain components are

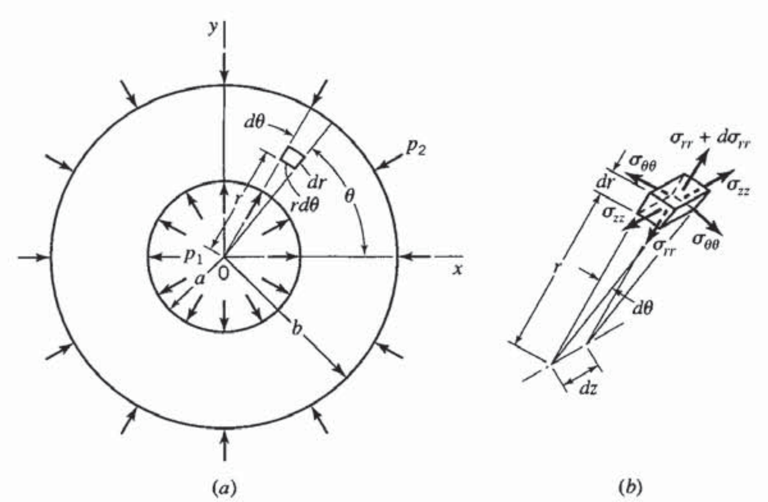


**FIGURE 11.3** Stresses in thick-wall cylinder. (*a*) Thin annulus of thickness $dz$. (The $z$ axis is perpendicular to the plane of the figure.) (*b*) Cylindrical volume element of thickness $dz$.

zero because of radial symmetry; furthermore, we assume that $\epsilon_{zz}$ is constant. Eliminating the displacement $u = u(r)$ from the first two of Eqs. 11.2, we obtain

$$r\frac{d\epsilon_{\theta\theta}}{dr} = \epsilon_{rr} - \epsilon_{\theta\theta} \quad \text{or} \quad \frac{d}{dr}(r\epsilon_{\theta\theta}) = \epsilon_{rr} \tag{11.3}$$

Equation 11.3 is the strain compatibility condition for the thick-wall cylinder.

### 11.1.3 Stress–Strain–Temperature Relations

The material of the cylinder is taken to be isotropic and linearly elastic. The stress–strain–temperature relations are (see Eqs. 3.38)

$$\begin{aligned}
\epsilon_{rr} &= \frac{1}{E}\left[\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right] + \alpha\Delta T \\
\epsilon_{\theta\theta} &= \frac{1}{E}\left[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right] + \alpha\Delta T \\
\epsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})\right] + \alpha\Delta T = \text{constant}
\end{aligned} \tag{11.4}$$

where $E$, $\nu$, and $\alpha$ denote the modulus of elasticity, Poisson's ratio, and the coefficient of linear thermal expansion, respectively. The term $\Delta T$ in Eq. 11.4 represents the change in temperature measured from a uniform reference temperature (constant throughout the cylinder initially); see Boresi and Chong (2000).

### 11.1.4 Material Response Data

For a cylinder made of isotropic linearly elastic material, the material response data are represented by the results of tests required to determine the elastic constants (modulus of elasticity $E$ and Poisson's ratio $\nu$) and the coefficient of linear thermal expansion $\alpha$. To determine the maximum elastic loads for the cylinder, the material data must include either the yield stress $Y$ obtained from a tension test or the shear yield stress $\tau_Y$ obtained from a torsion test of a hollow thin-wall tube.

## 11.2 STRESS COMPONENTS AT SECTIONS FAR FROM ENDS FOR A CYLINDER WITH CLOSED ENDS

In this section, we obtain expressions for the stress components $\sigma_{rr}$, $\sigma_{\theta\theta}$, $\sigma_{zz}$, for a cylinder with closed ends; the cylinder is subjected to internal pressure $p_1$, external pressure $p_2$, axial load $P$, and temperature change $\Delta T$ (Figure 11.1).

We may express Eq. 11.3 in terms of $\sigma_{rr}$, $\sigma_{\theta\theta}$, $\sigma_{zz}$ and their derivatives with respect to $r$, by substitution of the first two of Eqs. 11.4 into Eq. 11.3. Since $\epsilon_{zz}$ = constant, the last of Eqs. 11.4 may be used to express the derivative $d\sigma_{zz}/dr$ in terms of the derivatives of $\sigma_{rr}$, $\sigma_{\theta\theta}$, and $\Delta T$ with respect to $r$. By means of this expression, we may eliminate $d\sigma_{zz}/dr$ from Eq. 11.3 to rewrite Eq. 11.3 in terms of $\sigma_{rr}$, $\sigma_{\theta\theta}$, and derivatives of $\sigma_{rr}$, $\sigma_{\theta\theta}$, and $\Delta T$. Since the undifferentiated terms in $\sigma_{rr}$ and $\sigma_{\theta\theta}$ occur in the form $\sigma_{rr} - \sigma_{\theta\theta}$, Eq. 11.1 may be used to eliminate $\sigma_{rr} - \sigma_{\theta\theta}$. Hence, we obtain the differential expression

$$\frac{d}{dr}\left(\sigma_{rr} + \sigma_{\theta\theta} + \frac{\alpha E\,\Delta T}{1-\nu}\right) = 0 \tag{11.5}$$

Incorporated in Eq. 11.5 is the equation of equilibrium, Eq. 11.1, the strain compatibility equation, Eq. 11.3, and the stress–strain–temperature relations, Eqs. 11.4.

Integration of Eq. 11.5 yields the result

$$\sigma_{rr} + \sigma_{\theta\theta} + \frac{\alpha E\,\Delta T}{1-\nu} = 2C_1 \tag{11.6}$$

where $2C_1$ is a constant of integration (the factor 2 is included for simplicity of form in subsequent expressions). Elimination of the stress component $\sigma_{\theta\theta}$ between Eqs. 11.1 and 11.6 yields the following expression for $\sigma_{rr}$:

$$\frac{d}{dr}(r^2\sigma_{rr}) = -\frac{\alpha E\,\Delta T}{1-\nu} + 2C_1 r \tag{11.7}$$

Integration of Eq. 11.7 yields the result

$$\sigma_{rr} = -\frac{\alpha E}{r^2(1-\nu)}\int_a^r \Delta T r\,dr + \left(1 - \frac{a^2}{r^2}\right)C_1 + \frac{C_2}{r^2} \tag{11.8}$$

where the integration is carried out from the inner radius $a$ of the cylinder (Figure 11.1) to the radius $r$, and $C_2$ is a second constant of integration. Substitution of Eq. 11.8 into Eq. 11.6 yields the result

$$\sigma_{\theta\theta} = \frac{\alpha E}{r^2(1-\nu)}\int_a^r \Delta T r\,dr - \frac{\alpha E\,\Delta T}{1-\nu} + \left(1 + \frac{a^2}{r^2}\right)C_1 - \frac{C_2}{r^2} \tag{11.9}$$

By Eqs. 11.8 and 11.9, we obtain

$$\sigma_{rr} + \sigma_{\theta\theta} = 2C_1 - \frac{\alpha E\,\Delta T}{1-\nu} \tag{11.10}$$

Equation 11.10 serves as a check on the computations (see Eq. 11.6). The constants of integration $C_1$ and $C_2$ are obtained from the boundary conditions $\sigma_{rr} = -p_1$ at $r = a$ and $\sigma_{rr} = -p_2$ at $r = b$ (Figure 11.1). Substituting these boundary conditions into Eq. 11.8, we find

$$C_1 = \frac{1}{b^2 - a^2}\left(p_1 a^2 - p_2 b^2 + \frac{\alpha E}{1-\nu}\int_a^b \Delta T r\,dr\right), \quad C_2 = -p_1 a^2 \tag{11.11}$$

Hence, Eq. 11.10 may be written as

$$\frac{\sigma_{rr} + \sigma_{\theta\theta}}{2} = \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} - \frac{\alpha E \Delta T}{2(1-\nu)} + \frac{E}{(1-\nu)(b^2 - a^2)}\int_a^b \Delta\, T r\,dr \tag{11.12}$$

To obtain $\sigma_{zz}$, we integrate each term of the last of Eqs. 11.4 over the cross-sectional area of the cylinder. Thus, we have

$$\int_a^b \epsilon_{zz} 2\pi r dr = \frac{1}{E}\int_a^b \sigma_{zz} 2\pi r dr - \frac{2\nu}{E}\int_a^b \frac{\sigma_{\theta\theta} + \sigma_{rr}}{2} 2\pi r\, dr + \alpha \int_a^b \Delta T\, 2\pi r\, dr \tag{11.13}$$

For sections far removed from the end section, $\epsilon_{zz}$ is a constant, and the integral of $\sigma_{zz}$ over the cross-sectional area is equal to the applied loads. Hence, because of pressures $p_1$ and $p_2$ and axial load $P$ applied to an end plate (Figure 11.4), overall equilibrium in the axial direction requires

$$\int_a^b \sigma_{zz}\, 2\pi r\, dr = P + \pi(p_1 a^2 - p_2 b^2) \tag{11.14}$$

If there is no axial load $P$ applied to the closed ends, $P = 0$.

Since the temperature change $\Delta T$ does not appear in Eq. 11.14, the effects of temperature are self-equilibrating. With Eqs. 11.12–11.14, the expression for $\epsilon_{zz}$ at a section far removed from the ends can be written in the form

$$\epsilon_{zz(\text{closed end})} = \frac{1-2\nu}{E(b^2-a^2)}(p_1 a^2 - p_2 b^2) + \frac{P}{\pi(b^2-a^2)E} + \frac{2\alpha}{b^2-a^2}\int_a^b \Delta T r\, dr \tag{11.15}$$

Substitution of Eq. 11.15 into the last of Eqs. 11.4, with Eq. 11.12, yields the following expression for $\sigma_{zz}$ for a section far removed from the closed ends of the cylinder:

$$\sigma_{zz(\text{closed end})} = \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} + \frac{P}{\pi(b^2-a^2)} - \frac{\alpha E\, \Delta T}{1-\nu} + \frac{2\alpha E}{(1-\nu)(b^2-a^2)}\int_a^b \Delta T r\, dr \tag{11.16}$$

### 11.2.1 Open Cylinder

If a cylinder has open ends and there is no axial load applied on its ends, overall equilibrium of an axial portion of the cylinder (Figure 11.5) requires that

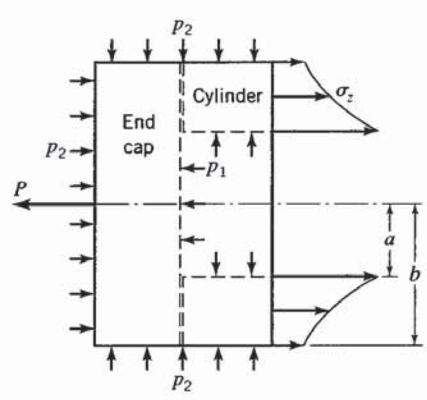


**FIGURE 11.4** Axial equilibrium of closed-end cylinder.

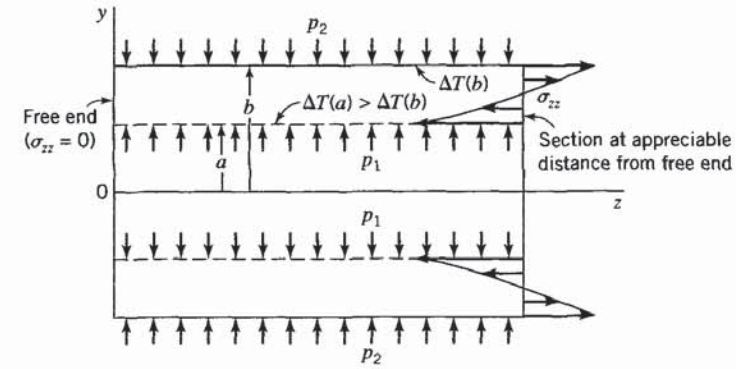


**FIGURE 11.5** Self-equilibrating axial stress distribution in an open cylinder.

$$\int_a^b 2\pi r \sigma_{zz}\, dz = 0 \tag{11.17}$$

Then, by Eqs. 11.12, 11.13, and 11.17 (also by Eqs. 11.14, 11.15, and 11.17), the expression for $\epsilon_{zz}$ may be written in the form

$$\epsilon_{zz(\text{open end})} = \frac{2\nu(p_2 b^2 - p_1 a^2)}{(b^2 - a^2)E} + \frac{2\alpha}{b^2 - a^2}\int_a^b \Delta T r\, dr \tag{11.18}$$

and for $\sigma_{zz}$, we obtain, by Eqs. 11.4, 11.12, and 11.18,

$$\sigma_{zz(\text{open end})} = \frac{\alpha E}{1 - \nu}\left(\frac{2}{b^2 - a^2}\int_a^b \Delta T r\, dr - \Delta T\right) \tag{11.19}$$

We note, by Eq. 11.19, that if the temperature change $\Delta T = 0$, then $\sigma_{zz} = 0$. However, $\epsilon_{zz} \neq 0$ (see Eq. 11.18) when the Poisson ratio $\nu \neq 0$. Note that if $p_1 = p_2 = P = 0$ (temperature change still occurs), Eqs. 11.15 and 11.16 are identical to Eqs. 11.18 and 11.19, respectively.

## 11.3 STRESS COMPONENTS AND RADIAL DISPLACEMENT FOR CONSTANT TEMPERATURE

### 11.3.1 Stress Components

In the absence of temperature change, we set $\Delta T = 0$. Then Eqs. 11.8–11.11 and 11.16 can be used to obtain the following expressions for the stress components in a closed cylinder (cylinder with end caps):

$$\sigma_{rr} = \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} - \frac{a^2 b^2}{r^2(b^2 - a^2)}(p_1 - p_2) \tag{11.20}$$

$$\sigma_{\theta\theta} = \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} + \frac{a^2 b^2}{r^2(b^2 - a^2)}(p_1 - p_2) \tag{11.21}$$

$$\sigma_{zz} = \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} + \frac{P}{\pi(b^2 - a^2)} = \text{constant} \tag{11.22}$$

$$\sigma_{rr} + \sigma_{\theta\theta} = \frac{2(p_1 a^2 - p_2 b^2)}{b^2 - a^2} = \text{constant} \tag{11.23}$$

For an open cylinder in the absence of axial force $P$, $\sigma_{zz} = 0$ by Eq. 11.19 with $\Delta T = 0$. Since the sum $\sigma_{rr} + \sigma_{\theta\theta}$ and stress $\sigma_{zz}$ are constants through the thickness of the wall of the closed cylinder, by Eq. 11.13 or 11.15, we see that $\epsilon_{zz}$ is constant (extension or compression).

### 11.3.2 Radial Displacement for a Closed Cylinder

For no temperature change, $\Delta T = 0$. Then the radial displacement $u$ for a point in a thick-wall closed cylinder (cylinder with end caps) may be obtained by the second of Eqs. 11.2, the second of Eqs. 11.4, and Eqs. 11.20–11.22. The resulting expression for $u$ is

$$u_{(\text{closed end})} = \frac{r}{E(b^2 - a^2)}\left[(1 - 2\nu)(p_1 a^2 - p_2 b^2) + \frac{(1+\nu)a^2 b^2}{r^2}(p_1 - p_2) - \nu\frac{P}{\pi}\right] \tag{11.24}$$

### 11.3.3 Radial Displacement for an Open Cylinder

Of special interest are open cylinders (cylinders without end caps), since an open inner cylinder is often shrunk to fit inside an open outer cylinder to increase the strength of the resulting composite cylinder. For an open cylinder, in the absence of temperature changes ($\Delta T = 0$), Eq. 11.19 yields $\sigma_{zz} = 0$. Hence, proceeding as for the closed cylinder, we obtain

$$u_{(\text{open end})} = \frac{r}{E(b^2 - a^2)}\left[(1 - \nu)(p_1 a^2 - p_2 b^2) + \frac{(1+\nu)a^2 b^2}{r^2}(p_1 - p_2)\right] \tag{11.25}$$

Equation 11.25 gives the radial displacement of any point at radius $r$ in an open cylinder. For internal pressure only, $p_2 = 0$. Then, by Eq. 11.25, the radial displacement at the inner surface $r = a$ is

$$u_a = \frac{a p_1}{E}\left(\frac{a^2 + b^2}{b^2 - a^2} + \nu\right) \tag{a}$$

Similarly, for external pressure only, $p_1 = 0$, and the radial displacement at the outer surface $r = b$ is

$$u_b = -\frac{bp_2}{E}\left(\frac{a^2+b^2}{b^2-a^2} - \nu\right) \tag{b}$$

**EXAMPLE 11.1**
***Stresses in a Hollow Cylinder***

A thick-wall cylinder is made of steel ($E = 200$ GPa and $\nu = 0.29$), has an inside diameter of 20 mm, and has an outside diameter of 100 mm. The cylinder is subjected to an internal pressure of 300 MPa. Determine the stress components $\sigma_{rr}$ and $\sigma_{\theta\theta}$ at $r = a = 10$ mm, $r = 25$ mm, and $r = b = 50$ mm.

**Solution**

The external pressure $p_2 = 0$. Equations 11.20 and 11.21 simplify to

$$\sigma_{rr} = p_1\frac{a^2(r^2-b^2)}{r^2(b^2-a^2)}$$

$$\sigma_{\theta\theta} = p_1\frac{a^2(r^2+b^2)}{r^2(b^2-a^2)}$$

Substitution of values for $r$ equal to 10 mm, 25 mm, and 50 mm, respectively, into these equations yields the following results:

| Stress | $r = 10$ mm | $r = 25$ mm | $r = 50$ mm |
|---|---|---|---|
| $\sigma_{rr}$ | -300.0 MPa | -37.5 MPa | 0.0 |
| $\sigma_{\theta\theta}$ | 325.0 MPa | 62.5 MPa | 25.0 MPa |

**EXAMPLE 11.2**
***Stresses and Deformations in a Hollow Cylinder***

A thick-wall closed-end cylinder is made of an aluminum alloy ($E = 72$ GPa and $\nu = 0.33$), has an inside diameter of 200 mm, and has an outside diameter of 800 mm. The cylinder is subjected to an internal pressure of 150 MPa. Determine the principal stresses, maximum shear stress at the inner radius ($r = a = 100$ mm), and the increase in the inside diameter caused by the internal pressure.

**Solution**

The principal stresses are given by Eqs. 11.20–11.22. For the conditions that $p_2 = 0$ and $r = a$, these equations give

$$\sigma_{rr} = p_1\frac{a^2-b^2}{b^2-a^2} = -p_1 = -150 \text{ MPa}$$

$$\sigma_{\theta\theta} = p_1\frac{a^2+b^2}{b^2-a^2} = 150\frac{100^2+400^2}{400^2-100^2} = 170 \text{ MPa}$$

$$\sigma_{zz} = p_1\frac{a^2}{b^2-a^2} = 150\frac{100^2}{400^2-100^2} = 10 \text{ MPa}$$

The maximum shear stress, given by Eq. 2.39, is

$$\tau_{max} = \frac{\sigma_{max}-\sigma_{min}}{2} = \frac{170-(-150)}{2} = 160 \text{ MPa}$$

The increase in the inside diameter caused by the internal pressure is equal to twice the radial displacement given by Eq. 11.24 for the conditions $p_2 = P = 0$ and $r = a$. Thus,

$$u_{(r=a)} = \frac{p_1 a}{E(b^2 - a^2)}\left[(1-2\nu)a^2 + (1+\nu)b^2\right]$$

$$= \frac{150(100)}{72{,}000(400^2 - 100^2)}\left[(1-0.66)100^2 + (1+0.33)400^2\right]$$

$$= 0.3003 \text{ mm}$$

The increase in the inside diameter caused by the internal pressure is 0.6006 mm.

## EXAMPLE 11.3 *Stresses in a Composite Cylinder*

Let the cylinder in Example 11.1 be a composite cylinder made by shrinking an outer cylinder onto an inner cylinder. Before assembly, the inner cylinder has inner and outer radii of $a = 10$ mm and $c_i = 25.072$ mm, respectively. Likewise, the outer cylinder has inner and outer radii of $c_o = 25.000$ mm and $b = 50$ mm, respectively. Determine the stress components $\sigma_{rr}$ and $\sigma_{\theta\theta}$ at $r = a = 10$ mm, $r = 25$ mm, and $r = b = 50$ mm for the composite cylinder. For assembly purposes, the inner cylinder is cooled to a uniform temperature $T_1$ and the outer cylinder is heated to a uniform temperature $T_2$ to enable the outer cylinder to slide freely over the inner cylinder. It is assumed that the two cylinders will slide freely if we allow an additional 0.025 mm to the required minimum difference in radii of 0.072 mm. Determine how much the temperature (in degrees Celsius) must be raised in the outer cylinder above the temperature in the inner cylinder to freely assemble the two cylinders. $\alpha = 0.0000117/°\text{C}$.

**Solution**

After the composite cylinder has been assembled, the change in stresses caused by the internal pressure $p_1 = 300$ MPa is the same as for the cylinder in Example 11.1. These stresses are added to the residual stresses in the composite cylinder caused by shrinking the outer cylinder onto the inner cylinder.

The initial difference between the outer radius of the inner cylinder and the inner radius of the outer cylinder is 0.072 mm. After the two cylinders have been assembled and allowed to cool to their initial uniform temperature, a shrink pressure $p_s$ is developed between the two cylinders. The pressure $p_s$ is an external pressure for the inner cylinder and an internal pressure for the outer cylinder. The magnitude of $p_s$ is obtained from the fact that the sum of the radial displacement of the inner surface of the outer cylinder and the radial displacement of the outer surface of the inner cylinder must equal 0.072 mm. Hence, by Eq. 11.25,

$$\frac{c_o}{E(b^2 - c_o^2)}\left[(1-\nu)p_s c_o^2 + (1+\nu)p_s b^2\right] - \frac{c_i}{E(c_i^2 - a^2)}\left[-(1-\nu)p_s c_i^2 - (1+\nu)p_s a^2\right] = 0.072$$

Solving for $p_s$, we obtain

$$p_s = 189.1 \text{ MPa}$$

The pressure $p_s$ produces stresses (residual or shrink-fit stresses) in the nonpressurized composite cylinder. For the inner and outer cylinders, the residual stresses $\sigma_{rr}^{R}$ and $\sigma_{\theta\theta}^{R}$ at the inner and outer radii are given by Eqs. 11.20 and 11.21. For the inner cylinder, $p_1 = 0$, $p_2 = p_s$, $a = 10$ mm, and $b = 25$ mm. For the outer cylinder, $p_1 = p_s$, $p_2 = 0$, $a = 25$ mm, and $b = 50$ mm. The residual stresses are found to be

| | Inner cylinder | | Outer cylinder | |
|---|---|---|---|---|
| **Residual stress** | $r = 10$ mm | $r = 25$ mm | $r = 25$ mm | $r = 50$ mm |
| $\sigma_{rr}^{R}$ | 0 | −189.1 MPa | −189.1 MPa | 0 |
| $\sigma_{\theta\theta}^{R}$ | −450.2 MPa | −261.1 MPa | 315.1 MPa | 126.0 MPa |

The stresses in the composite cylinder after an internal pressure of 300 MPa has been applied are obtained by adding these residual stresses to the stresses calculated in Example 11.1. Thus, we find

| | Inner cylinder | | Outer cylinder | |
|---|---|---|---|---|
| **Residual stress** | $r = 10$ mm | $r = 25$ mm | $r = 25$ mm | $r = 50$ mm |
| $\sigma_{rr}$ | −300.0 MPa | −226.6 MPa | −226.6 MPa | 0 |
| $\sigma_{\theta\theta}$ | −125.2 MPa | −198.6 MPa | 377.7 MPa | 151.0 MPa |

A comparison of the stresses for the composite cylinder with those for the solid cylinder in Example 11.1 indicates that the stresses have been changed greatly. The determination of possible improvements in the design of the open-end cylinder necessitates consideration of particular criteria of failure (see Section 11.4).

To have the inner cylinder slide easily into the outer cylinder during assembly, the difference in temperature between the two cylinders is given by the relation

$$\Delta T = T_2 - T_1 = \frac{u}{r\alpha} = \frac{0.072 + 0.025}{r\alpha} = \frac{0.097}{25(0.0000117)} = 331.6°\text{C}$$

since for uniform temperatures $T_1$, $T_2$, we have $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = 0$ in each cylinder, and since then Eqs. 11.2 and 11.4 yield $\epsilon_{\theta\theta} = u/r = \alpha \Delta T$, where $r = c_o = c_i$.

## 11.4 CRITERIA OF FAILURE

The criterion of failure used in the design of a thick-wall cylinder depends on the type of material in the cylinder. As discussed in Section 4.3, the maximum principal stress criterion should be used in the design of members made of brittle isotropic materials if the principal stress of largest magnitude is a tensile stress. Either the maximum shear-stress or the octahedral shear-stress criterion of failure should be used in the design of members made of ductile isotropic materials (see Section 4.4).

### 11.4.1 Failure of Brittle Materials

If a thick-wall cylinder is made of a brittle material, the material property associated with fracture is the ultimate tensile strength $\sigma_u$. At the failure loads, the maximum principal stress in the cylinder is equal to $\sigma_u$. If the maximum principal stress occurs at the constrained ends of the cylinder, it cannot be computed using the relations derived in Sections 11.2 and 11.3. At sections far removed from the ends, the maximum principal stress is either the circumferential stress $\sigma_{\theta\theta(r=a)}$ or the axial stress $\sigma_{zz}$. If the cylinder is loaded so that the magnitude of the maximum compressive principal stress is appreciably larger than the magnitude of the maximum tensile principal stress, the appropriate criterion of failure to be used in design is uncertain. Such conditions are not considered in this book.

### 11.4.2 Failure of Ductile Materials

If excessive elastic deformation is not a design factor, failure of members made of ductile materials may be initiated as the result of general yielding or fatigue. Failure of these members is predicted by either the maximum shear-stress criterion of failure or the octahedral shear-stress criterion of failure.

#### *General Yielding Failure*

Thick-wall cylinders that are subjected to static loads or peak loads only a few times during the life of the cylinder are usually designed for the general yielding limit state. General yielding may be defined to occur when yielding is initiated in the member at some point other than at a stress concentration. This definition is used in examples at the end of this section (see also Section 4.6). However, yielding may be initiated in the region of stress concentrations at the ends of the cylinder or at an opening for pipe connections. Yielding in such regions in usually highly localized and subsequent general yielding is unlikely. However, the possibility of failure by fatigue still may exist (see Chapter 16). General yielding sometimes is considered to occur only after the member has yielded over an extensive region, such as occurs with fully plastic loads. Fully plastic loads for thick-wall cylinders are discussed in Section 11.5.

#### *Fatigue Failure*

In practice, a thick-wall cylinder may be subjected to repeated pressurizations (loading and unloading) that may lead to fatigue failure. Since fatigue cracks often occur in the neighborhood of stress concentrations, every region of stress concentration must be considered in the design. In particular, the maximum shear stress must be determined in the region of stress concentrations, since fatigue cracking usually originates at a point where either the maximum shear stress or maximum octahedral shear stress occurs. The equations derived in Sections 11.2 and 11.3 cannot be used to compute the design stresses, unless the maximum stresses occur at sections of the cylinder far removed from end constraints or other stress concentration regions.

### 11.4.3 Material Response Data for Design

If a member fails by general yielding, the material property associated with failure is the yield stress. This places a limit either on the value of the maximum shear stress, if the maximum shear-stress criterion of failure is used, or on the value of the octahedral shear stress, if the octahedral shear-stress criterion of failure is used. If the member fails by fatigue, the material property associated with the failure is the fatigue strength. For high cycle fatigue, both the maximum shear-stress criterion of failure and octahedral shear-stress criterion of failure are used widely in conjunction with the fatigue strength (see Chapter 16, Example 16.1).

The yield stress and fatigue strength may be obtained by tests of either a tension specimen or hollow thin-wall tube. The values of these properties, as determined from tests of a hollow thin-wall tube in torsion, are found to lead to a more accurate prediction of the material response for thick-wall cylinders than the values obtained from a tension specimen. This is because the critical state of stress in the cylinder is usually at the inner wall of the cylinder, and for the usual pressure loading it is essentially one of pure shear (as occurs in the torsion test) plus a hydrostatic state of stress. Since in many materials a hydrostatic stress does not affect the yielding, the material responds (yields) as if it were subjected to a state of pure shear. Consequently, if the material properties are determined by means of a torsion test of a hollow thin-wall tube, the maximum shear-stress criterion and octahedral shear-stress criterion predict failure loads that differ by less than 1% for either closed or open cylinders. The difference in these predictions may be as much as 15.5% if the material properties are obtained from tension specimen tests (Section 4.4). These conclusions pertain in general to most metals. However, the yield of most plastics is influenced by the hydrostatic state of stress. Hence, for most plastics, these conclusions may not generally hold.

The deviatoric state of stress (see Section 2.4) in a closed cylinder is identical to that for pure shear. Hence, the maximum shear-stress and the octahedral shear-stress criteria of failure predict nearly identical factors of safety for the design of a closed cylinder if the yield stress for the material is obtained from torsion tests of hollow thin-wall tubes. Let the shear yield stress obtained from a torsion test of a thin-wall hollow tube specimen be designated as $\tau_Y$. If the maximum shear stress for the inner radius of a closed cylinder is set equal to $\tau_Y$, the pressure $p_Y$ required to initiate yielding is obtained. (The reader is asked to derive the formula for $p_Y$ in Problem 11.17.) For the special case of a closed cylinder with internal pressure only and with dimensions $b = 2a$, the yield pressure is found to be $p_Y = 0.75\ \tau_Y$; the corresponding dimensionless stress distribution is shown in Figure 11.6.

### 11.4.4 Ideal Residual Stress Distributions for Composite Open Cylinders

It is possible to increase the strength of a thick-wall cylinder by introducing beneficial residual stress distributions. The introduction of beneficial residual stresses can be accomplished in several ways. One method consists of forming a composite cylinder from two or more open cylinders. For example, in the case of two cylinders, the inner cylinder has an outer radius that is slightly larger than the inner radius of the outer cylinder. The inner cylinder is slipped inside the outer cylinder after first heating the outer cylinder and/or

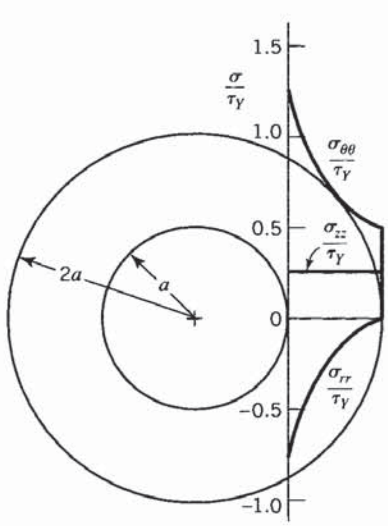


**FIGURE 11.6** Stress distributions in a closed cylinder at initiation of yielding ($b = 2a$).

cooling the inner cylinder. When the cylinders are allowed to return to their initially equal uniform temperatures (say, room temperature), a pressure (the so-called shrink pressure) is created between the cylinder surfaces in contact. This pressure introduces residual stresses in the cylinders. As a result, the strength of the composite cylinder under additional internal and external pressure loading is increased (Example 11.5). For more than two cylinders this process is repeated for each cylinder that is added to form the composite cylinder.

A second method for introducing residual stresses consists of pressurizing a single cylinder until it deforms inelastically to some distance into the wall from the inner surface (a process called autofrettage). When the pressure is removed, a beneficial residual stress distribution remains in the cylinder (see Section 11.5).

For a composite cylinder formed by two cylinders under a shrink fit and subject to internal pressure $p_1$, the most beneficial residual stress distribution is that which results in the composite cylinder failing (yielding or fracturing) simultaneously at the inner radii of the inner and outer cylinders. Consider, for example, a composite cylinder formed by inner and outer cylinders made of a *brittle* material whose stress–strain diagram remains linear up to its ultimate strength $\sigma_u$. The inner cylinder has inner radius $r_1$ and outer radius $1.5r_1$ + (i.e., the outer radius is slightly larger than $1.5r_1$). The outer cylinder has an inner radius of $1.5r_1$ and outer radius of $3r_1$. See Figure 11.7. Fracture of the brittle material occurs when the maximum principal stress reaches the ultimate strength $\sigma_u$. Since the maximum principal stress in the composite cylinder is the circumferential stress component $\sigma_{\theta\theta}$, for the most beneficial residual (dimensionless) stress distribution (Figure 11.7*a*), failure of the composite cylinder occurs when $\sigma_{\theta\theta} = \sigma_u$, simultaneously at the inner radii of the inner and outer cylinders (Figure 11.7*b*). The ideal residual stress distribution requires a specific difference between the inner radius of the outer cylinder and the outer radius of the inner cylinder, which produces a shrink pressure $p_s$ (see Problem 11.24). This shrink pressure produces a residual stress distribution (Figure 11.7*a*) such that the application of an internal pressure $p_1$ produces the (dimensionless) stress distribution of Figure 11.7*b* at failure.

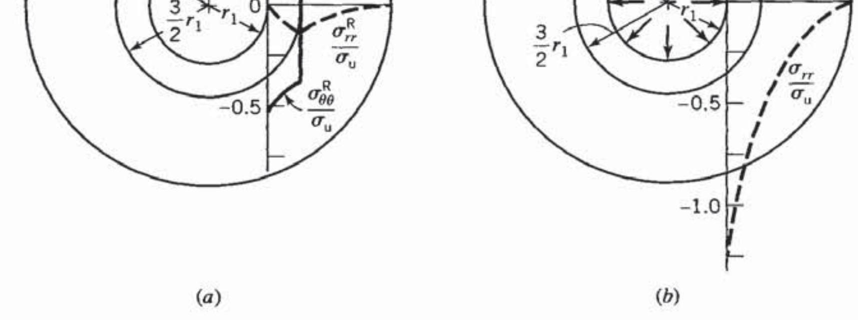


**FIGURE 11.7** Stress distributions in composite cylinder made of brittle material that fails at inner radius of both cylinders simultaneously. (*a*) Residual stress distributions. (*b*) Total stress distributions.

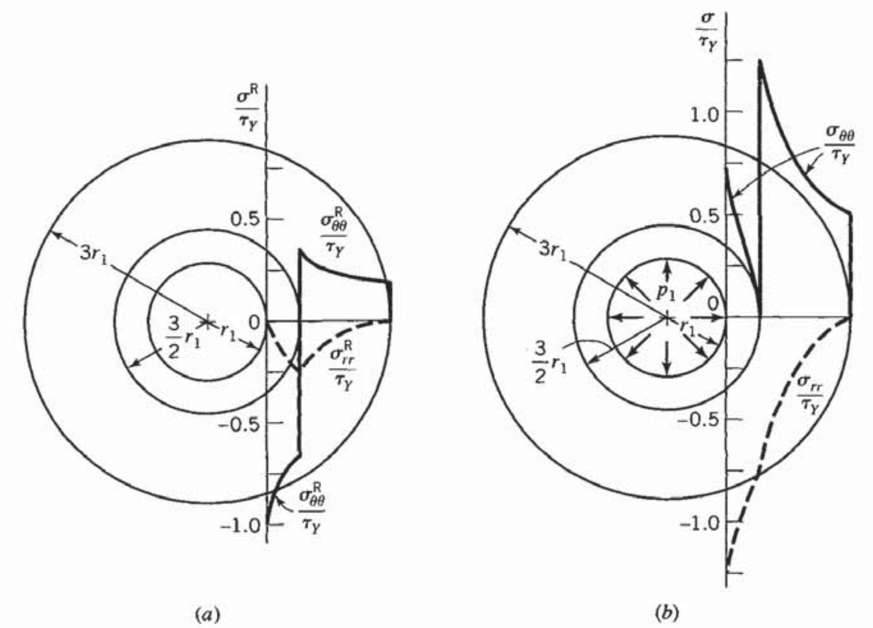


**FIGURE 11.8** Stress distributions in composite cylinder made of ductile material that fails at inner radius of both cylinders simultaneously. (*a*) Residual stress distributions. (*b*) Total stress distributions.

If the composite cylinder is made of a *ductile* metal, either the maximum shear-stress criterion of failure or the octahedral shear-stress criterion of failure can be used. For example, let the composite cylinder of Figure 11.8 be made of a ductile metal. Based on the maximum shear-stress criterion of failure, the ideal residual stress distribution resulting from the shrink pressure $p_s$ is shown in Figure 11.8*a*. (In this case, the interference fit is different from the cylinder of Figure 11.7; see Problem 11.23.) For an internal pressure $p_1$ at failure of the cylinder, yield occurs simultaneously at the inner radii of the inner and outer cylinders, and the associated dimensionless stress distribution is shown in Figure 11.8*b*.

**EXAMPLE 11.4** ***Yield Failure of Thick-Wall Cylinder***

The thick-wall cylinder in Example 11.1 is made of a ductile steel whose general yielding failure is accurately predicted by the octahedral shear-stress yield criterion. Determine the minimum yield stress for the steel for a factor of safety of $SF = 1.75$.

**Solution**

The stress components calculated in Example 11.1 are for a cylinder that has been designed with a factor of safety of $SF = 1.75$. Yielding impends in the cylinder when the internal pressure is increased to $(SF)p_1 = 525$ MPa. The yield stress $Y$ for the steel is obtained by setting the octahedral shear stress in the cylinder [when the pressure in the cylinder is $(SF)p_1$] equal to the octahedral shear stress that occurs in a tension specimen made of the steel when the tension specimen axial stress is $Y$. The octahedral shear stress in the tension specimen is given by the relation (see Eq. 2.22)

$$\tau_{oct} = \frac{1}{3}\sqrt{(Y-0)^2 + (0-0)^2 + (0-Y)^2} = \frac{\sqrt{2}Y}{3} \tag{a}$$

The octahedral shear stress at any point in the thick-wall cylinder is given by the relation (see Eq. 2.22)

$$\tau_{\text{oct}} = \frac{1}{3}\sqrt{(\sigma_{\theta\theta}-\sigma_{rr})^2 + (\sigma_{rr}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{\theta\theta})^2} \tag{b}$$

For the open cylinder, the axial stress $\sigma_{zz}$ is zero and the radial and circumferential stresses are

$$\sigma_{rr} = -1.75(300) = -525 \text{ MPa}$$

$$\sigma_{\theta\theta} = 1.75(325) = 568.8 \text{ MPa}$$

Substituting these stress components into Eq. (b) and setting Eq. (a) equal to Eq. (b), we obtain

$$Y = \frac{1}{\sqrt{2}}\sqrt{(568.8+525)^2 + (525)^2 + (568.8)^2} = 947.5 \text{ MPa}$$

## EXAMPLE 11.5 *Yield of a Composite Thick-Wall Cylinder*

The inner and outer cylinders of the composite thick-wall cylinder in Example 11.3 are made of the same ductile steel as the cylinder in Example 11.4. Determine the minimum yield stress for the steel in the composite cylinder for a factor of safety of $SF = 1.75$.

**Note:** Equations (a) and (b) in Example 11.4 are valid for this problem also.

### Solution

For the composite open cylinder, it is necessary to consider initiation of yielding for the inside of the inner cylinder, as well as for the inside of the outer cylinder. The axial stress $\sigma_{zz}$ is zero for both cylinders. At the inside of the inner cylinder, the radial and circumferential stresses for a pressure $(SF)p_1$ are

$$\sigma_{rr} = (1.75)(300) = -525 \text{ MPa}$$

$$\sigma_{\theta\theta} = (1.75)(325) - 450.2 = 118.6 \text{ MPa}$$

Substituting these stress components into Eq. (b) and setting Eq. (a) equal to Eq. (b), we obtain

$$Y = \frac{1}{\sqrt{2}}\sqrt{(118.6+525)^2 + (525)^2 + (118.6)^2} = 593.3 \text{ MPa}$$

At the inside of the outer cylinder, the radial and circumferential stresses for a pressure $(SF)p_1$ are

$$\sigma_{rr} = -(1.75)(37.5) - 189.1 = -254.7 \text{ MPa}$$

$$\sigma_{\theta\theta} = (1.75)(62.5) + 315.1 = 424.5 \text{ MPa}$$

Substituting these stress components into Eq. (b) and setting Eq. (a) equal to Eq. (b), we find

$$Y = \frac{1}{\sqrt{2}}\sqrt{(424.5+254.7)^2 + (254.7)^2 + (424.5)^2}$$

$$= 594.3 \text{ MPa} > 593.3 \text{ MPa}$$

For the composite cylinder, the yield stress should be at least $Y = 594.3$ MPa. An ideal design for a composite cylinder should cause the required yield stress to be the same for the inner and outer cylinders. (Note that the above design is nearly ideal.)

A comparison of the required yield stress for the single cylinder in Example 11.4 and the required yield stress for the composite cylinders indicates the advantage of the composite cylinder. The yield stress of the single cylinder material must be 59.4% greater than that of the composite cylinder, if both cylinders are subjected to the same initial pressure and are designed for the same factor of safety against initiation of yielding.

## 11.5 FULLY PLASTIC PRESSURE AND AUTOFRETTAGE

Thick-wall cylinders made of ductile material can be strengthened by introducing beneficial residual stress distributions. In Sections 11.3 and 11.4, it was found that beneficial residual stress distributions may be produced in a composite cylinder formed by shrinking one cylinder onto another. Beneficial residual stress distributions may also be introduced into a single cylinder by initially subjecting the cylinder to high internal pressure so that inelastic deformations occur in the cylinder. As a result, an increase in the load-carrying capacity of the cylinder occurs because of the beneficial residual stress distributions that remain in the cylinder after the high pressure is removed. The residual stress distribution in the unloaded cylinder depends on the depth of yielding produced by the high pressure, the shape of the inelastic portion of the stress–strain diagram for loading of a tensile specimen of the material, and the shape of the stress–strain diagram for unloading of the tensile specimen followed by compression loading of the specimen. If the material in the cylinder is a strain-hardening material, a part (usually, a small part) of the increase in load-carrying capacity is due to the strengthening of the material, resulting from strain hardening of the material. If the material exhibits a flat-top stress–strain diagram at the yield point (i.e., elastic–perfectly plastic), all the increase in load-carrying capacity is due to the beneficial residual stress distribution.

The process of increasing the strength of open and closed cylinders by increasing the internal pressure until the cylinder is deformed inelastically is called *autofrettage*. The beneficial effect of the autofrettage process increases rapidly with the spread of inelastic deformation through the wall thickness of the cylinder. Once yielding has spread through the entire wall thickness, any further improvement in load-carrying capacity resulting from additional inelastic deformation is due to strain hardening of the material. The minimum internal pressure $p_1$ required to produce yielding through the wall of the cylinder is an important pressure to be determined, since most of the increase in load-carrying capacity is produced below this pressure, and the deformation of the cylinder remains small up to this pressure. For the special case where the stress–strain diagram of the material is flat-topped at the yield point $Y$, the internal pressure $p_1$ is called the fully plastic pressure $p_P$.

We derive the fully plastic pressure by assuming that the maximum shear-stress criterion of failure is valid. If we assume that $\sigma_{zz}$ is the intermediate principal stress ($\sigma_{rr} < \sigma_{zz} < \sigma_{\theta\theta}$) for the cylinder, $\sigma_{\theta\theta} - \sigma_{rr} = 2\tau_Y$, where $\tau_Y$ is the shear yield stress. This result may be substituted into the equation of equilibrium, Eq. 11.1, to obtain

$$d\sigma_{rr} = \frac{2\tau_Y}{r}\,dr \tag{11.26}$$

Integration yields

$$\sigma_{rr} = 2\tau_Y \ln r + C \tag{11.27}$$

The constant of integration $C$ is obtained from the boundary condition that $\sigma_{rr} = -p_2$ when $r = b$. Thus, we obtain

$$\sigma_{rr} = -2\tau_Y \ln\frac{b}{r} - p_2 \tag{11.28}$$

which describes the radial stress distribution at the fully plastic pressure $p_P$. The magnitude of $p_P$ is given by Eq. 11.28 since the internal pressure is then $p_1 = p_P = -\sigma_{rr}$ at $r = a$. Thus, we obtain

$$p_P = 2\tau_Y \ln\frac{b}{a} + p_2 \tag{11.29}$$

In practice, $p_2$ is ordinarily taken equal to zero, since for $p_2 = 0$ the required internal pressure $p_1$ is smaller than for nonzero $p_2$. The circumferential stress distribution for the cylinder at the fully plastic pressure is obtained by substituting Eq. 11.28 into the relation $\sigma_{\theta\theta} - \sigma_{rr} = 2\tau_Y$ to obtain

$$\sigma_{\theta\theta} = 2\tau_Y\left(1 - \ln\frac{b}{r}\right) - p_2 \tag{11.30}$$

If the material in the cylinder is a *Tresca material*, that is, a material satisfying the maximum shear-stress criterion of failure, $\tau_Y = Y/2$, and the fully plastic pressure given by Eq. 11.29 is valid for cylinders subjected to axial loads in addition to internal and external pressures as long as $\sigma_{zz}$ is the intermediate principal stress, that is, $\sigma_{rr} < \sigma_{zz} < \sigma_{\theta\theta}$. If the material in the cylinder is a *von Mises material*, that is, a material satisfying the octahedral shear-stress criterion of failure, $\tau_Y = Y/\sqrt{3}$ (see column 4 of Table 4.2), the fully plastic pressure given by Eq. 11.29 is valid for closed cylinders subjected to internal and external pressures only. For this loading, the octahedral shear-stress criterion of failure requires that the axial stress be given by the relation

$$\sigma_{zz} = \frac{\sigma_{\theta\theta} + \sigma_{rr}}{2} \tag{11.31}$$

The proof of Eq. 11.31 is left to the reader.

In many applications, the external pressure $p_2$ is zero. In this case, the ratio of the fully plastic pressure $p_P$ (Eq. 11.29) to the pressure $p_Y$ that initiates yielding in the cylinder at the inner wall (see Problem 11.17) is given by the relation

$$\frac{p_P}{p_Y} = \frac{2b^2}{b^2 - a^2}\ln\frac{b}{a} \tag{11.32}$$

In particular, this ratio becomes large as the ratio $b/a$ becomes large. For $b = 2a$, Eq. 11.32 gives $p_P = 1.85p_Y$; dimensionless radial, circumferential, and axial stress distributions for this cylinder are shown in Figure 11.9. A comparison of these stress distributions with those at initiation of yielding (see Figure 11.6) indicates that yielding throughout the wall thickness of the cylinder greatly alters the stress distributions. If the cylinder in Figure 11.9 unloads elastically, the residual stress distributions can be obtained by multiplying the stresses in Figure 11.6 by the factor 1.85 and subtracting them from the stresses in Figure 11.9. For instance, the residual circumferential stress $\sigma^R_{\theta\theta}$ at the inner radius is calculated to be $\sigma^R_{\theta\theta} = -1.72\tau_Y$. This maximum circumferential residual stress can be expressed in terms of the tensile yield stress $Y$ as follows: for a Tresca material $\sigma^R_{\theta\theta} = -0.86Y$ and for a von Mises material $\sigma^R_{\theta\theta} = -0.99Y$. However, one cannot always rely on the presence of this large compressive residual stress in the unloaded cylinder. In particular, all metals behave inelastically (because of the Bauschinger effect) when the cylinder is unloaded, resulting in a decrease in the beneficial effects of the residual stresses. For

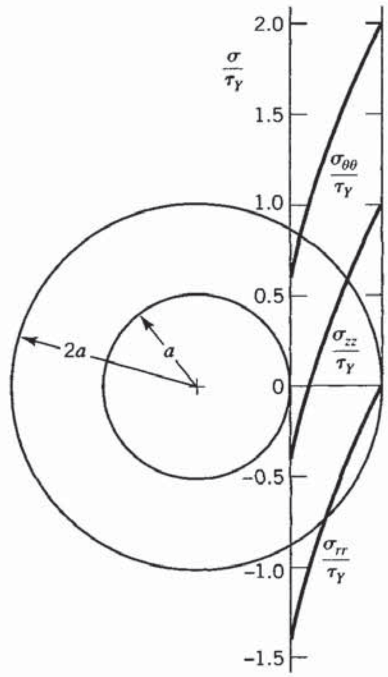


**FIGURE 11.9** Stress distributions in a closed cylinder made of a von Mises material at fully plastic pressure ($b = 2a$).

example, one investigation (Sidebottom et al., 1976) indicated that the beneficial effect of the residual stresses at the inside of the cylinder (when $b = 2a$) is decreased to about 50% of that calculated based on the assumption that the cylinder unloads elastically. Consequently, the cylinder will respond inelastically rather than elastically the next time it is loaded to the fully plastic pressure.

**EXAMPLE 11.6** ***Fully Plastic Pressure for a Cylinder***

A closed cylinder has an inner radius of 20 mm and an outer radius of 40 mm. It is made of steel that has a yield stress of $Y = 450$ MPa and obeys the von Mises yield criterion.

(a) Determine the fully plastic internal pressure $p_P$ for the cylinder.

(b) Determine the maximum circumferential and axial residual stresses when the cylinder is unloaded from $p_P$, assuming that the values based on linear elastic unloading are decreased by 50% because of inelastic deformation during unloading.

(c) Assuming that the elastic range of the octahedral shear stress has not been altered by the inelastic deformation, determine the internal pressure $p_1$ that can be applied to the cylinder based on a factor of safety $SF = 1.80$. For $SF = 1.80$, compare this result with the pressure $p_1$ for a cylinder without residual stresses.

**Solution**

(a) The shear yield stress $\tau_Y$ for the von Mises steel is obtained using the octahedral shear-stress yield criterion

$$\tau_Y = \frac{Y}{\sqrt{3}} = 259.8 \text{ MPa}$$

The magnitude of $p_P$ is given by Eq. 11.29. Thus, we find

$$p_P = 2\tau_Y \ln\frac{b}{a} = 2(259.8)\ln\frac{40}{20} = 360.21 \text{ MPa}$$

The circumferential and axial stresses at the inner radius for fully plastic conditions are given by Eqs. 11.30 and 11.31. They are

$$\sigma_{\theta\theta} = 2\tau_Y\left(1-\ln\frac{b}{a}\right) = 2(259.8)\left(1-\ln\frac{40}{20}\right) = 159.4\text{ MPa}$$

$$\sigma_{zz} = \frac{\sigma_{\theta\theta}+\sigma_{rr}}{2} = \frac{159.4-360.2}{2} = -100.4\text{ MPa}$$

**(b)** Assuming linearly elastic unloading, we compute the circumferential and axial residual stresses at $r = a$ as

$$\begin{aligned}\sigma_{\theta\theta}^{\mathrm{R}} &= 159.4-\frac{p_{\mathrm{P}}(b^2+a^2)}{b^2-a^2} = 159.4-\frac{360.2(40^2+20^2)}{40^2-20^2}\\ &= 440.9\text{ MPa}\\ \sigma_{zz}^{\mathrm{R}} &= -100.4-\frac{p_{\mathrm{P}}a^2}{b^2-a^2} = -100.4-\frac{360.2(20^2)}{40^2-20^2} = 220.5\text{ MPa}\end{aligned}$$

The actual residual stresses may be as much as 50% less than these computed values. Thus,

$$\begin{aligned}\sigma_{\theta\theta}^{\mathrm{R}} &= 0.50(-440.9) = -220.4\text{ MPa}\\ \sigma_{zz}^{\mathrm{R}} &= 0.50(-220.5) = -110.2\text{ MPa}\end{aligned} \tag{a}$$

**(c)** Yielding is initiated in the cylinder at a pressure $(SF)p_1 = 1.80p_1$. If the residual stresses are neglected, the stresses at the inner radius caused by pressure $(SF)p_1$ are

$$\begin{aligned}\sigma_{rr} &= -(SF)(p_1) = -1.80p_1\\ \sigma_{\theta\theta} &= (SF)(p_1)\frac{b^2+a^2}{b^2-a^2} = (1.80)(p_1)\frac{40^2+20^2}{40^2-20^2}\\ &= 3.000p_1\\ \sigma_{zz} &= (SF)(p_1)\frac{a^2}{b^2-a^2} = (1.80)(p_1)\frac{20^2}{40^2-20^2}\\ &= 0.6000p_1\end{aligned} \tag{b}$$

The actual stresses at the inner radius are obtained by adding the residual stresses given by Eq. (a) to those given by Eqs. (b). Thus,

$$\begin{aligned}\sigma_{rr} &= -1.80p_1\\ \sigma_{\theta\theta} &= 3.0000p_1-220.4\\ \sigma_{zz} &= 0.6000p_1-110.2\end{aligned} \tag{c}$$

The octahedral shear-stress yield condition requires that

$$\frac{\sqrt{2}Y}{3} = \frac{1}{3}\sqrt{(\sigma_{\theta\theta}-\sigma_{rr})^2+(\sigma_{rr}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{\theta\theta})^2} \tag{d}$$

Substituting the values for the stress components given by Eqs. (c) into Eq. (d), we find that

$$p_1 = 154.2 \text{ MPa}$$

is the working internal pressure for the cylinder that was preloaded to the fully plastic pressure. Substituting the values for the stress components given by Eqs. (b) into Eq. (d), we obtain the working internal pressure for the cylinder without residual stresses:

$$p_1 = 108.3 \text{ MPa}$$

Hence, the working pressure for the cylinder that is preloaded to the fully plastic pressure is 42.4% greater than the working pressure for the elastic cylinder without residual stresses.

## 11.6 CYLINDER SOLUTION FOR TEMPERATURE CHANGE ONLY

Consider the stress distribution in a thick-wall cylinder subjected to uniform internal and external pressures $p_1$ and $p_2$, axial load $P$, and temperature change $\Delta T$ that depends on the radial coordinate $r$ only. The stress distribution may be obtained from Eqs. 11.8–11.11 and 11.16. The special case of constant uniform temperature was considered in Section 11.3. In this section, the case of a cylinder subjected to a temperature change $\Delta T = T(r)$, in the absence of pressures and axial load, is treated. If internal and external pressures and temperature changes occur simultaneously, the resulting stresses may be obtained by superposition of the results of this section with those of Section 11.3. As in Section 11.3, the results here are restricted to the static, steady-state problem. Accordingly, the steady-state temperature change $\Delta T = T(r)$ is required input to the problem.

### 11.6.1 Steady-State Temperature Change (Distribution)

The temperature distribution in a homogeneous body in the absence of heat sources is given by Fourier's heat equation

$$\beta \nabla^2 T = \frac{\partial T}{\partial t} \tag{11.33}$$

in which $\beta$ is the thermal diffusivity for the material in the body, where we consider $T = \Delta T$ to be the temperature change measured from the uniform reference temperature of the unstressed state, and $t$ is the time. For steady-state conditions, $\partial T/\partial t = 0$, and Eq. 11.33 reduces to

$$\nabla^2 T = 0 \tag{11.34}$$

In cylindrical coordinates $(r, \theta, z)$, Eq. 11.34 takes the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{11.35}$$

Since $T$ is assumed to be a function of $r$ only, Eq. 11.35 simplifies to

$$\frac{d^2 T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \tag{11.36}$$

The solution of Eq. 11.36 is

$$T = C_1 \ln r + C_2 \tag{11.37}$$

where $C_1$ and $C_2$ are constants of integration. With Eq. 11.37, the boundary conditions $T = T_b$ for $r = b$ and $T = T_a$ for $r = a$ are used to determine $C_1$ and $C_2$. The solution of Eq. 11.37 then takes the form

$$T = \frac{T_0}{\ln\left(\frac{b}{a}\right)} \ln\frac{b}{r} \tag{11.38}$$

where

$$T_0 = T_a - T_b$$

### 11.6.2 Stress Components

If $p_1 = p_2 = P = 0$, Eq. 11.38 can be used with Eqs. 11.8–11.11 and 11.16 to obtain stress components for steady-state temperature distributions in a thick-wall cylinder. The results are

$$\sigma_{rr} = \frac{\alpha E T_0}{2(1-\nu)\ln\left(\frac{b}{a}\right)}\left[-\ln\frac{b}{r} + \frac{a^2(b^2-r^2)}{r^2(b^2-a^2)}\ln\frac{b}{a}\right] \tag{11.39}$$

$$\sigma_{\theta\theta} = \frac{\alpha E T_0}{2(1-\nu)\ln\left(\frac{b}{a}\right)}\left[1 - \ln\frac{b}{r} - \frac{a^2(b^2+r^2)}{r^2(b^2-a^2)}\ln\frac{b}{a}\right] \tag{11.40}$$

$$\sigma_{zz} = \sigma_{rr} + \sigma_{\theta\theta} = \frac{\alpha E T_0}{2(1-\nu)\ln\left(\frac{b}{a}\right)}\left[1 - 2\ln\frac{b}{r} - \frac{2a^2}{b^2-a^2}\ln\frac{b}{a}\right] \tag{11.41}$$

Thus, the stress distributions for linearly elastic behavior of a thick-wall cylinder subjected to a steady-state temperature distribution are given by Eqs. 11.39–11.41. When $T_0 = T_a - T_b$ is positive, the temperature $T_a$ at the inner radius is greater than the temperature $T_b$ at the outer radius. For the case of positive $T_0$, dimensionless stress distributions for a cylinder with $b = 2a$ are shown in Figure 11.10. For this case, the stress components $\sigma_{\theta\theta}$ and $\sigma_{zz}$ are compressive, so a positive temperature difference $T_0$ is beneficial for a cylinder that is subjected to a combination of internal pressure $p_1$ and temperature since the compressive stresses resulting from $T_0$ counteract tensile stresses resulting from $p_1$. The stresses in cylinders subjected to internal pressure $p_1$, external pressure $p_2$, axial load $P$, and steady-state temperature may be obtained as follows: The radial stress is given by adding Eq. 11.20 to Eq. 11.39, the circumferential stress is given by adding Eq. 11.21 to Eq. 11.40, and the axial stress is given by adding Eq. 11.22 to Eq. 11.41.

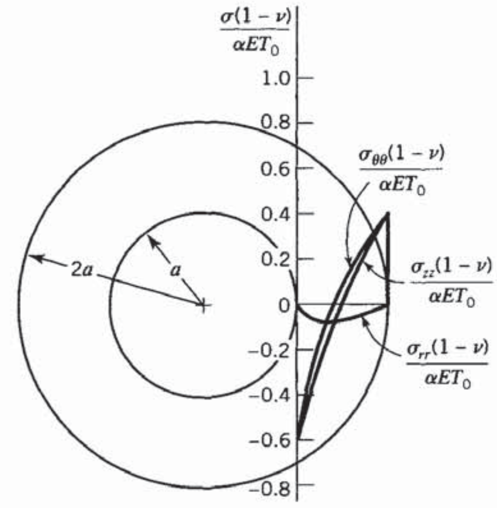


**FIGURE 11.10** Stress distributions in a cylinder subjected to a temperature gradient ($b = 2a$).

## 11.7 ROTATING DISKS OF CONSTANT THICKNESS

Consider a circular disk of inner radius $a$, outer radius $b$, and constant thickness $t << b$ (Figure 11.11$a$). Let the disk rotate with constant angular velocity $\omega$ [rad/s] about an axis perpendicular to its plane at 0. For axially symmetric plane stress ($\sigma_{zz} = 0$), the stress–strain relations in polar coordinates are (see Section 3.4)

$$\begin{aligned} \sigma_{rr} &= \frac{E}{1-\nu^2}(\epsilon_{rr} + \nu\epsilon_{\theta\theta}) - \frac{E\alpha T}{1-\nu} \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu^2}(\nu\epsilon_{rr} + \epsilon_{\theta\theta}) - \frac{E\alpha T}{1-\nu} \end{aligned} \tag{11.42}$$

where we let $T = \Delta T$ be the change in temperature from a reference state, $\alpha$ is the coefficient of thermal expansion, $E$ is the modulus of elasticity, and $\nu$ is Poisson's ratio. The strain components $\epsilon_{rr}$ and $\epsilon_{\theta\theta}$ are related to the radial displacement $u = u(r)$ by (see Eq. 11.2)

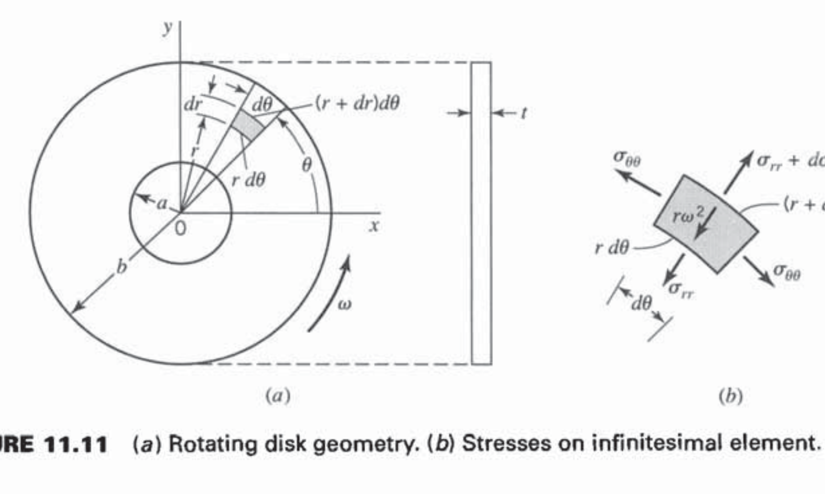


**FIGURE 11.11** ($a$) Rotating disk geometry. ($b$) Stresses on infinitesimal element.

$$\epsilon_{rr} = \frac{du}{dr}, \quad \epsilon_{\theta\theta} = \frac{u}{r} \tag{11.43}$$

Substitution of Eqs. 11.43 into Eqs. 11.42 yields

$$\begin{aligned} \sigma_{rr} &= \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) - \frac{E\alpha T}{1-\nu} \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu^2}\left(\nu\frac{du}{dr} + \frac{u}{r}\right) - \frac{E\alpha T}{1-\nu} \end{aligned} \tag{11.44}$$

Consider next the equilibrium condition for an element of the disk (Figure 11.11*b*). By equating the sum of forces in the radial $r$ direction to the mass times the acceleration of the element, we obtain

$$(\sigma_{rr} + d\sigma_{rr})(r + r\,d\theta)t - \sigma_{rr}(r\,d\theta)dt - 2\sigma_{\theta\theta}\left(dr\frac{d\theta}{2}\right)t = -\rho r\omega^2(r\,dr\,d\theta)t \tag{a}$$

where $\rho$ is the mass per unit volume and $r\omega^2$ is the radial acceleration of the element.

Neglecting higher-order terms in Eq. (a), we find

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = -\rho r\omega^2 \tag{11.45}$$

Substitution of Eqs. 11.44 into Eq. 11.45 yields

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = -\frac{(1-\nu^2)}{E}\rho r\omega^2 + (1+\nu)\frac{d(\alpha T)}{dr} \tag{b}$$

Rewriting the left side of Eq. (b), we have

$$\frac{d}{dr}\left[\frac{1}{r}\frac{du}{dr}(ru)\right] = -\frac{(1-\nu^2)}{E}\rho r\omega^2 + (1+\nu)\frac{d}{dr}(\alpha T) \tag{11.46}$$

Direct integration of Eq. 11.46 yields

$$u = -\frac{(1-\nu^2)}{8E}\rho r^3\omega^2 + \frac{\alpha(1+\nu)}{r}\int rT\,dr + C_1 r + \frac{C_2}{r} \tag{11.47}$$

where $C_1$ and $C_2$ are constants of integration. The constants $C_1$ and $C_2$ are determined by the boundary conditions at $r = a$ and $r = b$ (Figure 11.11*a*). For example, with no forces applied at $r = a$ and $r = b$, we have

$$\sigma_{rr} = 0 \quad \text{at } r = a \quad \text{and} \quad r = b \tag{11.48}$$

Hence, by Eqs. 11.44 and Eq. 11.47, with $T = 0$, we find

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left[-\frac{(3+\nu)(1-\nu^2)}{8E}\rho r^2\omega^2 + (1+\nu)C_1 - \frac{(1-\nu)}{r^2}C_2\right] \tag{11.49}$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left[-\frac{(1+3\nu)(1-\nu^2)}{8E}\rho r^2\omega^2 + (1+\nu)C_1 + \frac{(1-\nu)}{r^2}C_2\right] \tag{11.50}$$

Substituting Eq. 11.49 into Eqs. 11.48 and solving for $C_1$ and $C_2$, we obtain

$$\begin{aligned} C_1 &= \frac{(3+\nu)(1-\nu^2)}{8(1+\nu)E}(a^2+b^2)\rho\omega^2 \\ C_2 &= \frac{(3+\nu)(1-\nu^2)}{8(1-\nu)E}a^2b^2(\rho\omega^2) \end{aligned} \tag{11.51}$$

Then by Eqs. 11.49–11.51, we get

$$\begin{aligned} \sigma_{rr} &= \frac{3+\nu}{8}\rho b^2\omega^2\left[1 + \frac{a^2-r^2}{b^2} - \frac{a^2}{r^2}\right] \\ \sigma_{\theta\theta} &= \frac{3+\nu}{8}\rho b^2\omega^2\left[1 + \frac{(3+\nu)a^2-(1+3\nu)r^2}{(3+\nu)b^2} + \frac{a^2}{r^2}\right] \end{aligned} \tag{11.52}$$

We see by the first of Eqs. 11.52 that $\sigma_{rr} = 0$ for $r = a$ and $r = b$. Also, $\sigma_{rr}$ takes on a maximum value at $r = \sqrt{(ab)}$ (where $d\sigma_{rr}/dr = 0$). This maximum value is given by

$$(\sigma_{rr})_{\max} = \frac{3+\nu}{8}\rho b^2\omega^2\left(1-\frac{a}{b}\right)^2 \tag{11.53}$$

By the second of Eqs. 11.52, we find that $\sigma_{\theta\theta}$ is a maximum at $r = a$ (at the inner edge of the disk), where

$$(\sigma_{\theta\theta})_{\max} = \frac{3+\nu}{4}\rho b^2\omega^2\left(1 + \frac{1-\nu}{3+\nu}\frac{a^2}{b^2}\right) \tag{11.54}$$

Hence, at the inner edge, $(\sigma_{\theta\theta})_{\max}$ varies parabolically as a function of $a/b$.

By Eqs. 11.53 and 11.54, we see that $(\sigma_{\theta\theta})_{\max} > (\sigma_{rr})_{\max}$ for all values of $a$ and $b$. Also, by the second of Eqs. 11.52, we see that as $a/b \to 0$, there is a very large increase in $\sigma_{\theta\theta}$ near the inner edge of the disk (as $r \to a$). Thus as $a/b \to 0$ and $r \to a$, by the second of Eqs. 11.52, we obtain

$$(\sigma_{\theta\theta})_{\max} = \frac{3+\nu}{4}\rho b^2\omega^2 \tag{11.55}$$

Equation 11.55 indicates that the stress $\sigma_{\theta\theta}$ is increased due to the stress concentration of a small hole at the center of the disk.

Alternatively, as $a \to b$ (as the disk becomes a thin ring), the second of Eq. 11.51 gives

$$(\sigma_{\theta\theta})_{\max} \to \rho b^2\omega^2 \tag{11.56}$$

This analysis shows that the stresses produced in rotating disks, for example in rotors of electrical generators and gas turbine engines, are proportional to the square of the peripheral velocity $b\omega$. This fact limits the diameter of the disk or rotor so as not to exceed the material strength or working stress limits.

**EXAMPLE 11.7** ***Rotating Solid Disk***

Consider a solid disk of radius $b$ subjected to an angular velocity $\omega$ (Figure E11.7).

**(a)** Determine the polar coordinate stresses $\sigma_{rr}$ and $\sigma_{\theta\theta}$ in the disk as functions of $\rho$, $\nu$, $r$, $b$, and $\omega$. Let $T = 0$.

**(b)** For temperature change $T = 0$, determine the maximum values of $\sigma_{rr}$ and $\sigma_{\theta\theta}$ and their locations.

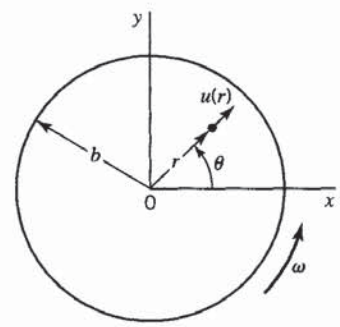


**FIGURE E11.7**

**Solution**

**(a)** The boundary conditions for the disk are

$$
\begin{aligned}
u &= 0 \quad \text{at} \quad r = 0 \\
\sigma_{rr} &= 0 \quad \text{at} \quad r = b
\end{aligned}
\tag{a}
$$

By Eq. 11.47 the general solution for the rotating disk is

$$
u = -\frac{(1-\nu^2)}{8E}\rho r^3\omega^2 + C_1 r + \frac{1}{r}C_2 \tag{b}
$$

By Eqs. 11.49 and 11.50, the stresses are

$$
\begin{aligned}
\sigma_{rr} &= \frac{E}{1-\nu^2}\left[-\frac{(3+\nu)(1-\nu^2)}{8E}\rho r^2\omega^2 + (1+\nu)C_1 - \frac{(1-\nu)}{r^2}C_2\right] \\
\sigma_{\theta\theta} &= \frac{E}{1-\nu^2}\left[-\frac{(1+3\nu)(1-\nu^2)}{8E}\rho r^2\omega^2 + (1+\nu)C_1 + \frac{(1-\nu)}{r^2}C_2\right]
\end{aligned}
\tag{c}
$$

Hence, by the first of Eqs. (a) and Eq. (b),

$$
C_2 = 0 \tag{d}
$$

Likewise, by Eq. (b) with $C_2 = 0$, the first of Eqs. (c) and the second of Eqs. (a) yield

$$
C_1 = \frac{(3+\nu)(1-\nu)}{8E}\rho b^2\omega^2 \tag{e}
$$

Consequently, by Eqs. (c)–(e), we obtain the stresses

$$\sigma_{rr} = \frac{(3+\nu)}{8}\rho b^2\omega^2\left[1 - \frac{r^2}{b^2}\right]$$
$$\sigma_{\theta\theta} = \frac{(3+\nu)}{8}\rho b^2\omega^2\left[1 - \frac{(1+3\nu)}{(3+\nu)}\frac{r^2}{b^2}\right] \tag{f}$$

(**b**) Since $r \leq b$, then by Eqs. (f) $\sigma_{rr}$ and $\sigma_{\theta\theta}$ are both positive and increase as $r \rightarrow 0$. Hence, at $r = 0$, the stresses approach their maximum values

$$(\sigma_{rr})_{\max} = (\sigma_{\theta\theta})_{\max} = \frac{3+\nu}{8}\rho b^2\omega^2 \tag{g}$$

Comparing Eq. (g) to Eq. 11.55, we see that the maximum stress in a solid disk, which occurs at its center, is one-half as large as the maximum stress resulting from the stress concentration at the edge of a small hole at the center of a disk. In other words, the stress concentration factor of the small hole is 2.0.

## EXAMPLE 11.8 *Plastic Deformation of a Rotating Disk*

A circular steel disk of inner radius $a = 100$ mm and outer radius $b = 300$ mm is subjected to a constant angular velocity $\omega$ [rad/s] (Figure E11.8). The steel has material properties $Y = 620$ MPa, $E = 200$ GPa, $\nu = 0.29$, and $\rho = 7.85 \times 10^3$ kg/m$^3$. Assume that the disk is in a state of plane stress ($\sigma_{zz} = 0$) and that yield is governed by the maximum shear-stress criterion. Also, let the disk be traction free at $r = a$ and $r = b$, and let $T = 0$.

(**a**) Determine the angular velocity $\omega_Y$ at which yield in the disk is initiated.

(**b**) Determine the angular velocity $\omega_P$ at which the disk is fully plastic; compare $\omega_P$ to $\omega_Y$.

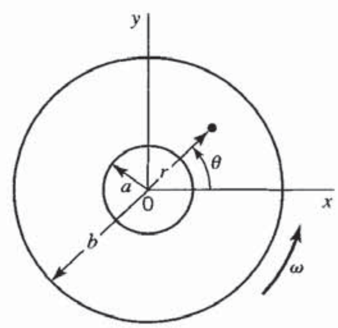


**FIGURE E11.8**

**Solution**

(**a**) Since $\sigma_{zz} = 0$ and $\sigma_{\theta\theta} > \sigma_{rr}$ for all values of $a$ and $b$ (see Eqs. 11.49 and 11.50), then the maximum shear-stress criterion is given by

$$\tau_{\max} = \frac{1}{2}Y = \frac{1}{2}(\sigma_{\theta\theta} - \sigma_{zz})_{\max} \tag{a}$$

or

$$(\sigma_{\theta\theta})_{\max} = Y = 620 \text{ MPa} \tag{b}$$

Since the disk is traction free at $r = a$ and $r = b$, the maximum value of $\sigma_{\theta\theta}$ is given by Eq. 11.54 as

$$(\sigma_{\theta\theta})_{max} = \frac{3+\nu}{4}\rho b^2\omega^2\left(1 + \frac{1-\nu}{3+\nu}\frac{a^2}{b^2}\right) \tag{c}$$

With the given data, Eqs. (b) and (c) yield

$$\omega_Y = \sqrt{\frac{4Y}{\rho[(3+\nu)b^2 + (1-\nu)a^2]}} \tag{d}$$

or

$$\omega_Y = 1021 \text{ rad/s} \tag{e}$$

**(b)** When the angular speed increases beyond $\omega_Y$, a plastic zone develops at $r = a$ (Figure E11.8). In other words, the region $a < r < r_P$ will be plastic and the region $r_P < r < b$ will be elastic, where $r = r_P$ is the interface between the plastic and elastic regions. In the plastic region $\sigma_{\theta\theta} = Y$, but $\sigma_{rr}$ is not known. However, we may obtain $\sigma_{rr}$ from the equilibrium condition (Eq. 11.45). Thus, with $\sigma_{\theta\theta} = Y$, we have in the plastic region

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\sigma_{rr} = \frac{Y}{r} - \rho r\omega^2$$

or, rewriting the left side, we have

$$\frac{1}{r}\frac{d}{dr}(r\sigma_{rr}) = \frac{Y}{r} - \rho r\omega^2$$

Integration yields

$$\sigma_{rr} = Y - \frac{1}{3}\rho r^2\omega^2 + \frac{C}{r} \tag{f}$$

For $r = a$, $\sigma_{rr} = 0$; so by Eq. (f),

$$C = \frac{1}{3}\rho a^3\omega^2 - aY \tag{g}$$

Equations (f) and (g) yield

$$\sigma_{rr} = Y\left(1 - \frac{a}{r}\right) - \frac{1}{3}\rho\omega^2\left(r^2 - \frac{a^3}{r}\right) \tag{h}$$

When the disk is fully plastic, $r = r_P = b$. Then, since $\sigma_{rr} = 0$ at $r = b$, Eq. (h) yields with the given data

$$\omega = \omega_P = \sqrt{\frac{3Y}{\rho(b^2 + ab + a^2)}} = 1350 \text{ rad/s} \tag{i}$$

Comparing $\omega_P$ to $\omega_Y$, we have by Eqs. (e) and (i)

$$\frac{\omega_P}{\omega_Y} = 1.32$$

In this case the speed at the fully plastic condition is 32% larger than that at yield.

**EXAMPLE 11.9** ***Residual Stresses in a Disk***

For the disk of Example 11.8, after the fully plastic state is reached, the angular velocity is reduced to zero. Determine the residual stresses $(\sigma_{rr})_R$ and $(\sigma_{\theta\theta})_R$ in the disk.

**Solution**

As noted in Section 6.10, the residual stresses may be obtained by subtracting the elastic stresses using $\omega = \omega_P$ from the fully plastic stresses. By Eqs. 11.52 with $\omega = \omega_P$, the elastic stresses are

$$
\begin{aligned}
(\sigma_{rr})_E &= \frac{3+\nu}{8}\rho\omega_P^2\left(a^2+b^2-r^2-\frac{a^2b^2}{r^2}\right)\\
(\sigma_{\theta\theta})_E &= \frac{1}{8}\rho\omega_P^2\left[(3+\nu)\left(a^2+b^2+\frac{a^2b^2}{r^2}\right)-(1+3\nu)r^2\right]
\end{aligned}
\tag{a}
$$

By Example 11.8, the plastic stresses are

$$
\begin{aligned}
(\sigma_{rr})_P &= Y\left(1-\frac{a}{r}\right)-\frac{1}{3}\rho\omega^2\left(r^2-\frac{a^3}{r}\right)\\
(\sigma_{\theta\theta})_P &= Y
\end{aligned}
\tag{b}
$$

Hence, by Eqs. (a) and (b),

$$
\begin{aligned}
(\sigma_{rr})_R &= (\sigma_{rr})_P-(\sigma_{rr})_E\\
&= Y\left(1-\frac{a}{r}\right)-\frac{1}{3}\rho\omega^2\left(r^2-\frac{a^3}{r}\right)-\left\{\frac{3+\nu}{8}\rho\omega_P^2\left(a^2+b^2-r^2-\frac{a^2b^2}{r^2}\right)\right\}\\
(\sigma_{\theta\theta})_R &= (\sigma_{\theta\theta})_P-(\sigma_{\theta\theta})_E\\
&= Y-\frac{1}{8}\rho\omega^2\left[(3+\nu)\left(a^2+b^2+\frac{a^2b^2}{r^2}\right)-(1+3\nu)r^2\right]
\end{aligned}
\tag{c}
$$

Since $\omega_P^2 = 3Y/[\rho(b^2+ab+a^2)]$, we see by the first of Eqs. (c) that $(\sigma_{rr})_R = 0$ for $r = a$ and $r = b$. Also, by eliminating $\omega_P$ from Eqs. (c), we may express the residual stresses in terms of $Y$ as follows:

$$
\begin{aligned}
(\sigma_{rr})_R &= Y\left\{\frac{r-a}{r}-\frac{3}{b^2+ab+a^2}\left[\frac{r^3-a^3}{3r}+\frac{3+\nu}{8}\left(a^2+b^2-r^2-\frac{a^2b^2}{r^2}\right)\right]\right\}\\
(\sigma_{\theta\theta})_R &= Y\left\{1-\frac{3}{8(b^2+ab+a^2)}\left[(3+\nu)\left(a^2+b^2+\frac{a^2b^2}{r^2}\right)-(1+3\nu)r^2\right]\right\}
\end{aligned}
\tag{d}
$$

Hence, with $a = 100$ mm, $b = 300$ mm, and $\nu = 0.29$, Eqs. (d) yield

$$
\begin{aligned}
\frac{(\sigma_{rr})_R}{Y} &= 0.05096154+1.7980769r^2-\frac{0.09230769}{r}+\frac{0.00854135}{r^2}\\
\frac{(\sigma_{\theta\theta})_R}{Y} &= 0.05096154+5.39423077r^2-\frac{0.00854135}{r^2}
\end{aligned}
\tag{e}
$$

As a check, the first of Eqs. (e) yields

$$\text{for } r = 100 \text{ mm}, \quad \frac{(\sigma_{rr})_R}{Y} = 0$$

$$\text{for } r = 300 \text{ mm}, \quad \frac{(\sigma_{rr})_R}{Y} = 0$$

Values of the residual stresses are given in Table E11.9 and a plot is shown in Figure E11.9. Since $\sigma_{zz} = 0$, the maximum value of shear stress (see Eqs. 4.14 and 4.15) does not cause yielding upon unloading; that is, $|(\sigma_{rr})_R - \sigma_{zz}| < Y$, $|(\sigma_{\theta\theta})_R - \sigma_{zz}| < Y$, and $|(\sigma_{\theta\theta})_R - (\sigma_{rr})_R| < Y$.

**TABLE E11.9 Residual Stress Ratios**

| $r$ | $(\sigma_{rr})_R/Y$ | $(\sigma_{\theta\theta})_R/Y$ | $\lvert(\sigma_{\theta\theta})_R/Y - (\sigma_{rr})_R/Y\rvert$ |
|---|---|---|---|
| 0.10 | 0.00000 | −0.74923 | 0.74923 |
| 0.11 | −0.06055 | −0.58966 | 0.52912 |
| 0.12 | −0.09923 | −0.46451 | 0.36528 |
| 0.13 | −0.12331 | −0.36328 | 0.23998 |
| 0.14 | −0.13735 | −0.27909 | 0.14174 |
| 0.15 | −0.14435 | −0.20728 | 0.06293 |
| 0.16 | −0.14628 | −0.14459 | 0.00169 |
| 0.17 | −0.14451 | −0.08869 | 0.05582 |
| 0.18 | −0.13998 | −0.03789 | 0.10209 |
| 0.19 | −0.13336 | 0.00909 | 0.14245 |
| 0.20 | −0.12512 | 0.05320 | 0.17832 |
| 0.21 | −0.11562 | 0.09517 | 0.21079 |
| 0.22 | −0.10512 | 0.13557 | 0.24069 |
| 0.23 | −0.09380 | 0.17485 | 0.26865 |
| 0.24 | −0.08180 | 0.21338 | 0.29518 |
| 0.25 | −0.06923 | 0.25144 | 0.32067 |
| 0.26 | −0.05617 | 0.28926 | 0.34543 |
| 0.27 | −0.04267 | 0.32704 | 0.36971 |
| 0.28 | −0.02879 | 0.36492 | 0.39372 |
| 0.29 | −0.01456 | 0.40305 | 0.41762 |
| 0.30 | 0.00000 | 0.44154 | 0.44154 |

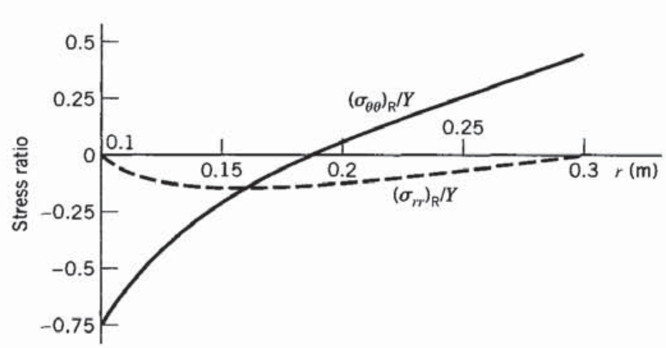


**FIGURE E11.9** Residual stress distribution.

# PROBLEMS

### Section 11.3

**11.1.** For the hollow cylinder of Example 11.1, determine the radial displacements at the inner surface and the outer surface.

**11.2.** For the hollow cylinder of Example 11.2, determine the principal stresses and the maximum shear stress at the outer surface and the increase of the outer diameter.

**11.3.** An open thick-wall cylinder of inner radius $a = 100$ mm and outer radius $b = 200$ mm is subjected to an internal pressure $p_1 = 200$ MPa.

**a.** Determine the stress components $\sigma_{rr}$ and $\sigma_{\theta\theta}$ at $r = 100$ mm, $r = 150$ mm, and $r = 200$ mm.

**b.** Sketch the distribution of $\sigma_{rr}$ and $\sigma_{\theta\theta}$ through the wall of the cylinder.

**11.4.** A long closed cylinder has an internal radius $a = 100$ mm and an external radius $b = 250$ mm. It is subjected to an internal pressure $p_1 = 80.0$ MPa ($p_2 = 0$). Determine the maximum radial, circumferential, and axial stresses in the cylinder.

**11.5.** Determine the radial and circumferential stress distributions for the cylinder in Problem 11.4.

**11.6.** Consider a 1-m length of the unloaded cylinder in Problem 11.4 at a location in the cylinder some distance from the ends. What are the dimensions of this portion of the cylinder after $p_1 = 80.0$ MPa is applied? The cylinder is made of a steel for which $E = 200$ GPa and $\nu = 0.29$.

**11.7.** A closed cylinder has an inside diameter of 20 mm and an outside diameter of 40 mm. It is subjected to an external pressure $p_2 = 40$ MPa and an internal pressure of $p_1 = 100$ MPa. Determine the axial stress and circumferential stress at the inner radius.

**11.8.** A composite cylinder has inner radius $a$, outer radius $b$, and interface radius $c$ (Figure P11.8). Initially, the outer radius of the inner cylinder is larger than the inner radius of the outer cylinder by an amount $\delta$. Show that after assembly the shrink-fit pressure is

$$p_s = \frac{E\delta}{c}\left[\frac{(b^2-c^2)(c^2-a^2)}{2c^2(b^2-a^2)}\right]$$

where $E$ is the modulus of elasticity of the cylinders and $\delta/c$ is the shrinkage factor. *Hint:* The increase of the inner radius of the outer cylinder plus the decrease in the outer radius of the inner cylinder produced by $p_s$ must be equal to $\delta$. (See the solution of Example 11.3.)

**11.9.** In Problem 11.8 (Figure P11.8) let $a = 100$ mm, $c = 200$ mm, and $b = 300$ mm. For steel cylinders ($E = 200$ GPa) and a shrinkage factor $\delta/c = 0.001$, determine the shrinkage-fit stresses $\sigma_{\theta\theta}$ at $r = 100$ mm, $r = 150$ mm, $r = 250$ mm, and $r = 300$ mm.

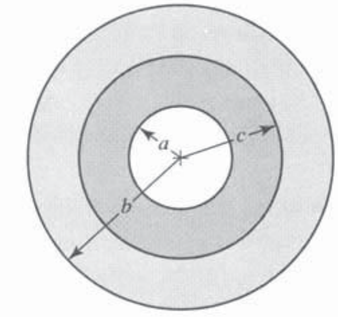


**FIGURE P11.8**

**11.10.** An aluminum composite cylinder ($E = 72$ GPa and $\nu = 0.33$) is made by shrinking an outer cylinder onto an inner cylinder (Figure P11.10). Initially the outer radius $c$ of the inner cylinder is larger than the inner radius of the outer cylinder by an amount $\delta = 0.125$ mm (see Problem 11.8). The cylinder is subjected to an internal pressure $p_i = 200$ MPa. Determine the stress $\sigma_{\theta\theta}$ in the inner cylinder at $r = 150$ mm and in the outer cylinder at $r = 150$ mm.

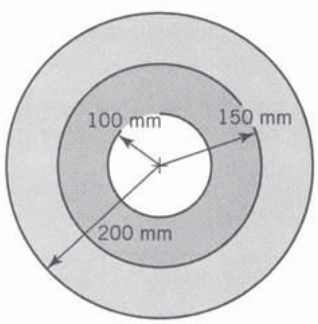


**FIGURE P11.10**

**11.11.** A composite aluminum alloy ($E = 72.0$ GPa and $\nu = 0.33$) cylinder is made up of an inner cylinder with inner and outer diameters of 80 and 120+ mm, respectively, and an outer cylinder with inner and outer diameters of 120 and 240 mm, respectively. The composite cylinder is subjected to an internal pressure of 160 MPa. What must the outside diameter of the inner cylinder be if the circumferential stress at the inside of the composite cylinder is equal to 130 MPa?

**11.12.** What must the outside diameter of the inner cylinder be for the composite cylinder in Problem 11.11 if the maximum shear stress at the inner radius of the inner cylinder is equal to the maximum shear stress at the inner radius of the outer cylinder? What are the values for the circumferential stress at the inside of the composite cylinder and the maximum shear stress?

**11.13.** A gray cast iron ($E$ = 103 GPa and $\nu$ = 0.20) cylinder has an outside diameter of 160 mm and an inside diameter of 40 mm. Determine the circumferential stress at the inner radius of the cylinder when the internal pressure is 60.0 MPa.

**11.14.** Let the cast iron cylinder in Problem 11.13 be a composite cylinder made up of an inner cylinder with inner and outer diameters of 40 and 80+ mm, respectively, and an outer cylinder with inner and outer diameters of 80 and 160 mm, respectively. What must the outside diameter of the inner cylinder be if the circumferential stress at the inside of the inner cylinder is equal to the circumferential stress at the inside of the outer cylinder? What is the magnitude of the circumferential stress at the inside of the composite cylinder?

**11.15.** A hollow steel hub ($E$ = 200 GPa and $\nu$ = 0.3), with an inner diameter of 100 mm and an outer diameter of 300 mm, is press-fitted over a solid steel shaft of diameter 100.125 mm. Determine the maximum principal stress in the shaft and in the hub. Ignore the stress concentration at the junction between the vertical sides of the hub and the shaft. See Figure P11.15.

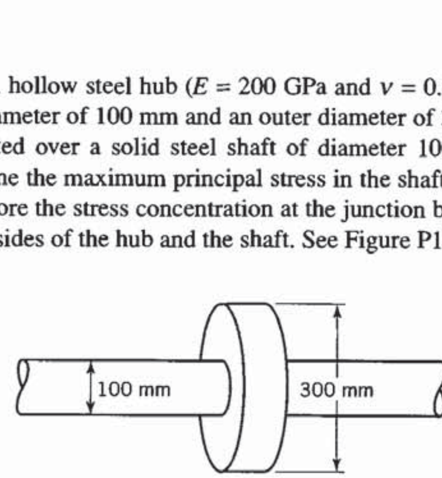


**FIGURE P11.15**

## Section 11.4

**11.16.** **a.** Derive the expression for the maximum shear stress in a thick-wall cylinder subjected to internal pressure $p_1$, external pressure $p_2$, and axial load $P$, assuming that $\sigma_{zz}$ is the intermediate principal stress, that is, $\sigma_{rr} < \sigma_{zz} < \sigma_{\theta\theta}$.

**b.** Derive an expression for the limiting value of the axial load $P$ for which the expression in part (a) is valid.

**11.17.** Let $\sigma_{zz}$ be the intermediate principal stress in a thick-wall cylinder ($\sigma_{rr} < \sigma_{zz} < \sigma_{\theta\theta}$). Using the maximum shear-stress criterion of failure, derive an expression for the internal pressure $p_Y$ necessary to initiate yielding in the cylinder. The shear yield stress for the material is $\tau_Y$.

**11.18.** For a closed cylinder subjected to internal pressure $p_1$ only, show that the octahedral shear stress $\tau_{\text{oct}}$ at the inner radius is given by the relation

$$\tau_{\text{oct}} = \frac{\sqrt{2}p_1 b^2}{\sqrt{3}(b^2 - a^2)}$$

**11.19.** A closed cylinder is made of a ductile steel that has a yield stress $Y$ = 600 MPa. The inside diameter of the cylinder is 80 mm. Determine the outside diameter of the cylinder if the cylinder is subjected to an internal pressure of $p_1$ = 140 MPa and the cylinder is designed using a factor of safety of $SF$ = 1.75 based on the maximum shear-stress criterion of failure.

**11.20.** Solve Problem 11.19 using the octahedral shear-stress criterion of failure.

**11.21.** A closed cylinder with inner and outer radii of 60 and 80 mm, respectively, is subjected to an internal pressure $p_1$ = 30.0 MPa and an axial load $P$ = 650 kN. The cylinder is made of a steel that has a yield stress of $Y$ = 280 MPa. Determine the factor of safety $SF$ used in the design of the cylinder based on

**a.** the maximum shear-stress criterion of failure and

**b.** the octahedral shear-stress criterion of failure

**11.22.** A closed cylinder with inner and outer diameters of 30 and 60 mm, respectively, is subjected to an internal pressure only. The cylinder is made of a brittle material having an ultimate strength of $\sigma_u$ = 160 MPa. The outer diameter has been gradually reduced as we move away from each end so that stress concentrations at the ends can be neglected. Determine the magnitude of $p_1$ based on a factor of safety of $SF$ = 3.00.

**11.23.** Two cylinders are slip-fitted together to form a composite open cylinder. Both cylinders are made of a steel having a yield stress $Y$ = 700 MPa. The inner cylinder has inner and outer diameters of 100 and 150+ mm, respectively. The outer cylinder has inner and outer diameters of 150 and 300 mm, respectively.

**a.** Determine the shrink pressure $p_s$ and maximum internal pressure $p_1$ that can be applied to the cylinder if it has been designed with a factor of safety of $SF$ = 1.85 for simultaneous initiation of yielding at the inner radii of the inner and outer cylinders. Use the maximum shear-stress criterion of failure.

**b.** Determine the outer diameter of the inner cylinder required for the design. For the steel $E$ = 200 GPa and $\nu$ = 0.29.

**11.24.** Two cylinders are slip-fitted together to form a composite open cylinder. Both cylinders are made of a brittle material whose stress–strain diagram is linear up to the ultimate strength $\sigma_u$ = 480 MPa. The inner cylinder has inner and outer radii of 50 and 75+ mm, respectively. The outer cylinder has inner and outer radii of 75 and 150 mm, respectively. Determine the shrink pressure $p_s$ and maximum internal pressure $p_1$ that results in initiation of fracture simultaneously at the inner radii of both cylinders. Use the maximum principal stress criterion of failure.

## Section 11.5

**11.25.** A thick-wall cylinder has an inside diameter of 180 mm and an outside diameter of 420 mm. It is made of steel having a yield stress of $Y$ = 460 MPa and obeying the Tresca criterion. Determine the fully plastic pressure for the cylinder if $p_2 = 0$.

**11.26.** **a.** Determine the working pressure $p_1$ for the thick-wall cylinder in Problem 11.25 if it is designed with a factor of safety of $SF = 3.00$ based on the fully plastic pressure.

**b.** What is the factor of safety based on the maximum elastic pressure $p_Y$?

**11.27.** A composite open cylinder has an inner cylinder with inner and outer radii of 20 and 30 mm, respectively, and is made of a steel with yield stress $Y_1 = 400$ MPa. The outer cylinder has inner and outer radii of 30 and 60 mm, respectively, and is made of a steel with yield stress $Y_2 = 600$ MPa. Determine the fully plastic pressure for the composite cylinder if both steels obey the von Mises criterion.

**11.28.** The closed cylinder in Example 11.6 is made of a Tresca material instead of a von Mises material. Obtain the solution for the Tresca material.

## Section 11.6

**11.29.** An unloaded closed cylinder has an inner radius of 100 mm and an outer radius of 250 mm. The cylinder is made of a steel for which $\alpha = 0.0000117/°C$, $E = 200$ GPa, and $\nu = 0.29$. Determine the stress components at the inner radius for a steady-state temperature change with the temperature at the inner radius 100°C greater than the temperature at the outer radius.

**11.30.** Let the steel in the cylinder in Problem 11.29 have a yield stress of $Y = 500$ MPa. Determine the magnitude of $T_0$ necessary to initiate yielding in the cylinder based on the

**a.** maximum shear-stress criterion of failure and

**b.** octahedral shear-stress criterion of failure

**11.31.** The cylinder in Problem 11.29 is subjected to a temperature difference of $T_0 = 50°C$ and an internal pressure $p_1 = 100$ MPa. Determine the stress components at the inner radius.

**11.32.** A closed brass ($Y = 240$ MPa, $E = 96.5$ GPa, $\nu = 0.35$, and $\alpha = 0.000020/°C$) cylinder has an inside diameter of 70 mm and an outside diameter of 150 mm. It is subjected to a temperature difference $T_0 = T_a - T_b = 70°C$.

**a.** Determine the magnitude $p_1$ of internal pressure required to initiate yield in the cylinder.

**b.** Determine the magnitude $p_2$ of external pressure required to initiate yield.

**c.** Repeat parts a and b for the case $T_0 = 0$. Use the maximum shear-stress criterion of failure.

## Section 11.7

**11.33.** A cast iron disk has an inner radius $a = 150$ mm and an outer radius $b = 300$ mm, with material properties $\rho = 7200\ kg/m^3$, $E = 70$ GPa, $\nu = 0.25$, and ultimate strength $\sigma_u = 170$ MPa. Determine the speed of revolution (in rpm) of the disk at which the maximum stress is equal to the ultimate strength.

**11.34.** In the proof test of a grinding wheel, the rotational speed is increased until the wheel bursts. The grinding wheel is a disk of inner radius $a = 100$ mm and an outer radius of $b = 400$ mm. The wheel is bonded to a steel shaft at the inner radius and has material properties $\rho = 2000\ kg/m^3$, $E = 12$ GPa, $\nu = 0.32$, and ultimate strength $\sigma_u = 20$ MPa. Determine the allowable rotational speed of the wheel using a safety factor of 2.0. Assume that the steel shaft is rigid and that there are no forces acting on the wheel at the outer radius.

**11.35.** A disk of inner radius $a$ and outer radius $b$ is subjected to an angular velocity $\omega$. The disk is constrained at $r = a$, so that the radial displacement $u$ is zero. At $r = b$, the disk is free of applied forces. Derive formulas for the constants of integration $C_1$ and $C_2$ (see Eqs. 11.47, 11.49, and 11.50), as functions of $\omega$, the material properties $\rho$, $E$, and $\nu$ and the radii $a$ and $b$. Note that with $C_1$ and $C_2$, the displacement $u$ and stresses $\sigma_{rr}$ and $\sigma_{\theta\theta}$ are also given as functions of $r$, etc.

**11.36.** A solid disk of radius $b$ is subjected to an angular velocity $\omega$ [rad/s]. The disk has mass density $\rho$, modulus of elasticity $E$, Poisson's ratio $\nu$, and yield strength $Y$. Temperature effects are negligible.

**a.** Determine the angular velocity $\omega_Y$ at which the disk yields initially. Assume that $\sigma_{zz} = 0$ and that the maximum shear-stress criterion applies.

**b.** Determine the angular velocity $\omega_P$ at which the disk becomes fully plastic. Compare $\omega_P$ to $\omega_Y$.

**c.** After the disk becomes fully plastic, it is returned to rest. Determine the resulting residual stresses in the disk.

**11.37.** The solid disk of Example 11.7 is subjected to an angular velocity $\omega$ [rad/s] and is also exposed to a temperature change

$$T = T_0\left(1 - \frac{r}{b}\right)$$

where $T_0$ is a positive constant.

**a.** Determine the additional stresses $\sigma_{rr}^T$ and $\sigma_{\theta\theta}^T$ resulting from $T$ as functions of $T_0$, $b$, and $r$.

**b.** Determine whether or not the stresses at $r = 0$ and $r = b$ are increased because of $T$.

**11.38.** A solid steel disk in a state of plane stress ($\sigma_{zz} = 0$) has an outer radius $b = 400$ mm and has properties $\rho = 7850\ kg/m^3$, $E = 200$ GPa, $\nu = 0.29$, and $Y = 620$ MPa. The disk is subjected to an angular velocity $\omega$ [rad/s], where $\omega_Y \le \omega \le \omega_P$, $\omega_Y$ is the angular velocity at yield, and $\omega_P$ is the angular velocity at which the disk is fully plastic (see Example 11.8). When $\omega$ increases beyond $\omega_Y$, a plastic zone develops at the center and progresses to $r = r_P$. As $\omega$ increases to $\omega_P$, the disk becomes fully plastic, that is, $r_P \to b$. Determine the location $r = r_P$ of the interface between the elastic and plastic regions as a function of $\omega$. *Hint:* Continuity of stresses at $r = r_P$ requires that

$$\sigma_{rr}(r_P^+) = \sigma_{rr}(r_P^-)$$

$$\sigma_{\theta\theta}(r_P^+) = \sigma_{\theta\theta}(r_P^-)$$

where at $r = r_P$, $r_P^+$ is in the elastic region and $r_P^-$ is in the plastic region. Also, the stresses in the elastic region are given by Eqs. 11.49 and 11.50.

**11.39.** A thin solid disk of radius $b$ rotates with angular velocity $\omega$ [rad/s] about an axis perpendicular to the disk at its center $r = 0$. It is also subjected to a temperature field $T = T_0 r/b$, where $T_0$ is a constant.

**a.** Determine the stresses $\sigma_{rr}$ and $\sigma_{\theta\theta}$ in the disk in terms of $T_0$ and $\omega$.

**b.** Determine the increase of its radius $b$ in terms of $T_0$ and $\omega$.

**11.40.** In Problem 11.38, let the disk have a central hole of radius $a = 100$ mm. Determine the value of $\omega$ for which the elastic–plastic interface occurs at the mean radius $r_P = (a + b)/2 = 250$ mm.

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